COMPLEX ANALYSIS FOR MATHEMATICS AND ENGINEERING

JOHN H. MATHEWS
RUSSELL W. HOWELL

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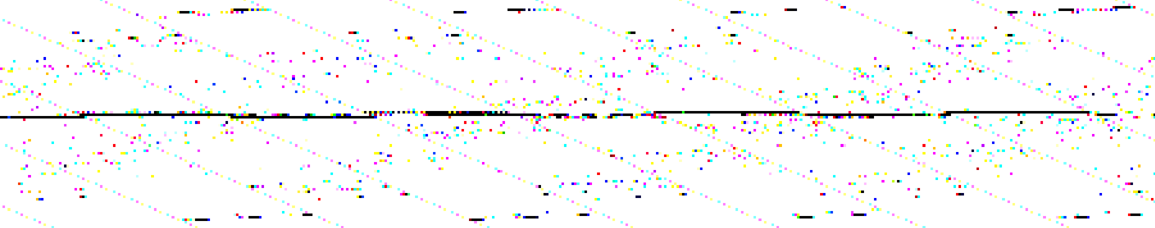

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2.2 Transformations and Linear Mappings

We now take our first look at the geometric interpretation of a complex function. If D is the domain of definition of the real-valued functions $u(x, y)$ and $v(x, y)$, then the system of equations

$$(1) \quad u = u(x, y) \quad \text{and} \quad v = v(x, y)$$

describes a transformation or mapping from D in the xy plane into the uv plane. Therefore, the function

$$(2) \quad w = f(z) = u(x, y) + iv(x, y)$$

can be considered as a mapping or transformation from the set D in the z plane onto the range R in the w plane. This is illustrated in Figure 2.1.

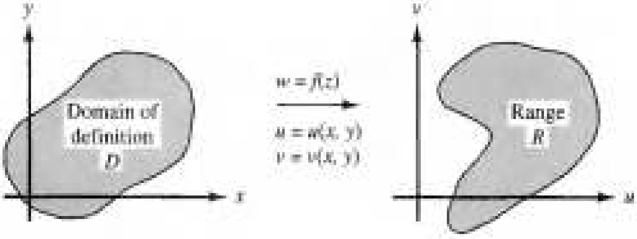

FIGURE 2.1 The mapping $w = f(z)$.

If A is a subset of the domain of definition D, then the set $B = \{f(z): z \in A\}$ is called the *image* of the set A, and f is said to *map A onto B*. The image of a single point is a single point, and the image of the entire domain D is the range R. The mapping $w = f(z)$ is said to be *from A into S* if the image of A is contained in S. The inverse image of a point w is the set of all points z in D such that $w = f(z)$. The inverse image of a point may be one point, several points, or none at all. If the latter case occurs, then the point w is not in the range of f.

The function f is said to be *one-to-one* if it maps distinct points $z_1 \neq z_2$ onto distinct points $f(z_1) \neq f(z_2)$. If $w = f(z)$ maps the set A one-to-one and onto the set B, then for each w in B there exists exactly one point z in A such that $w = f(z)$. Then loosely speaking, we can solve the equation $w = f(z)$ by solving for z as a function of w. That is, the *inverse function* $z = g(w)$ can be found, and the following equations hold:

$$(3) \quad \begin{aligned} &g(f(z)) = z \quad \text{for all } z \text{ in } A \quad \text{and} \\ &f(g(w)) = w \quad \text{for all } w \text{ in } B. \end{aligned}$$

Conversely, if $w = f(z)$ and $z = g(w)$ are functions that map A into B and B into A, respectively, and equations (3) hold, then $w = f(z)$ maps the set A one-to-one and onto the set B. The one-to-one property is easy to show, for if we have $f(z_1) = f(z_2)$, then $g(f(z_1)) = g(f(z_2))$; and using equation (3), we obtain $z_1 = z_2$.

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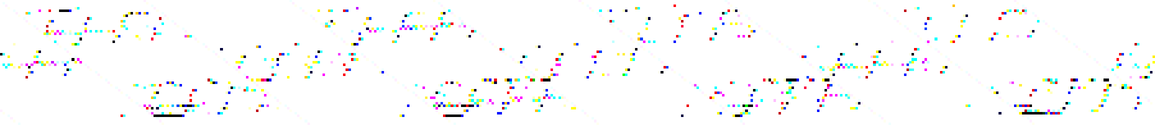

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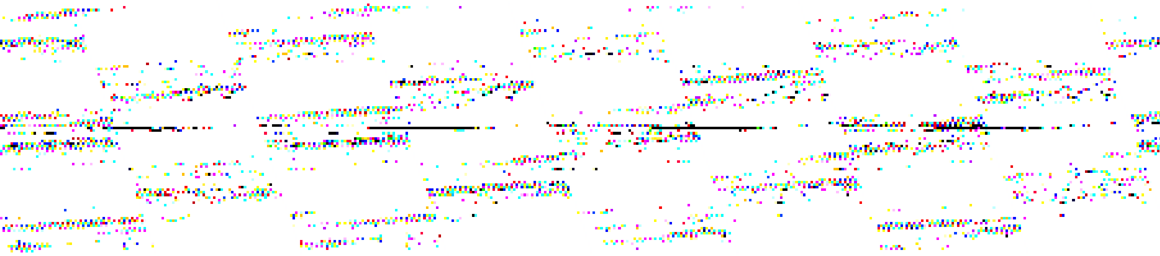

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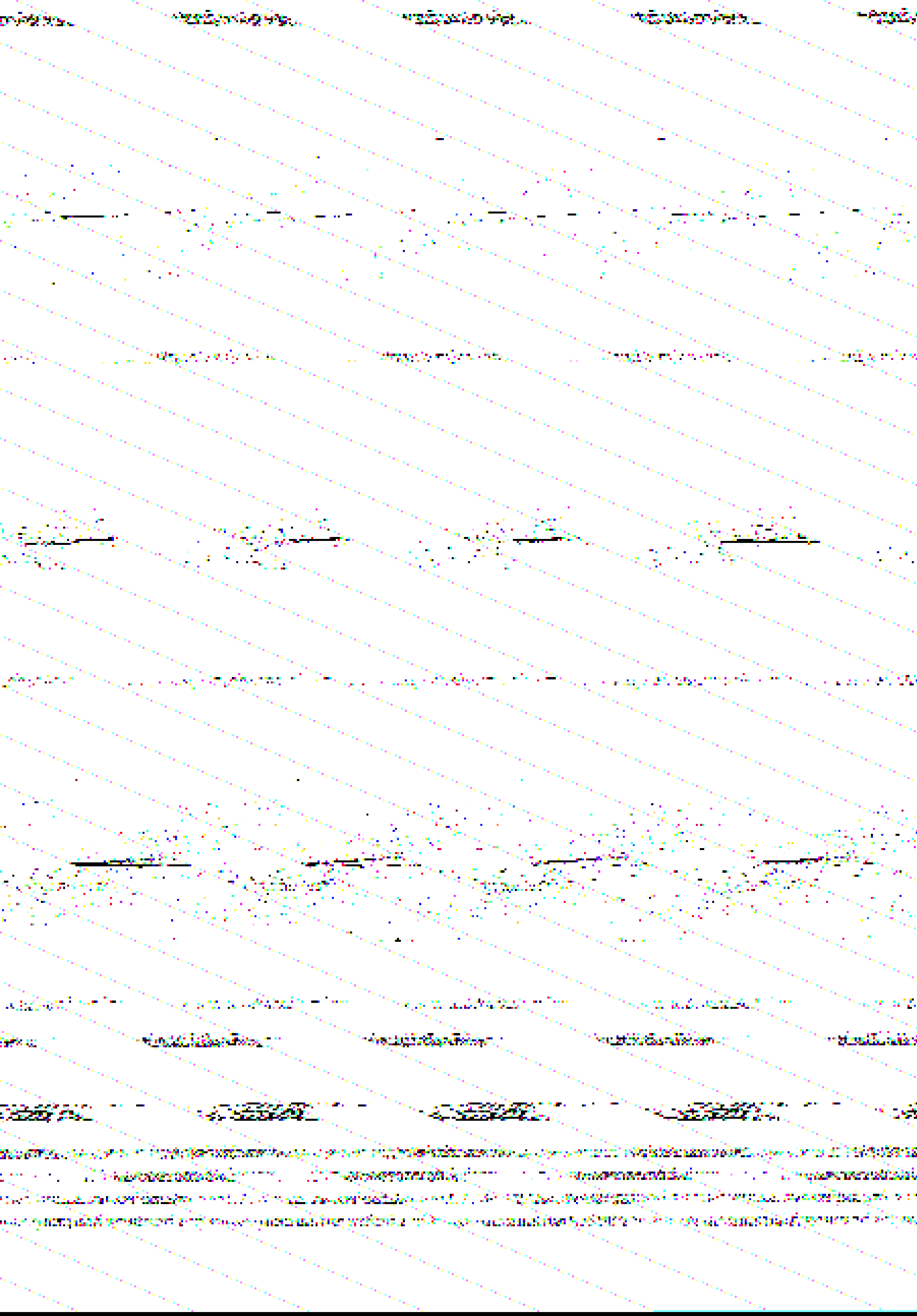

An ε-neighborhood of the point at infinity is the set $\{z: |z| > 1/\varepsilon\}$. The usual way to visualize the point at infinity is accomplished by using the stereographic projection and is attributed to Riemann. Let Ω be a sphere of diameter 1 that is centered at the $(0, 0, \frac{1}{2})$ in the three-dimensional space where coordinates are denoted by the triple of real numbers (x, y, ξ). Here the complex number $z = x + iy$ will be associated with the point $(x, y, 0)$.

The point $\mathcal{N} = (0, 0, 1)$ on Ω is called the *north pole* of Ω. Let z be a complex number, and consider the line segment L in three-dimensional space that joins z to the north pole $\mathcal{N}$. Then L intersects Ω in exactly one point $\mathcal{L}$. The correspondence $z \leftrightarrow \mathcal{L}$ is called the stereographic projection of the complex z plane onto the *Riemann sphere* Ω. A point $z = x + iy$ of unit modulus will correspond to $\mathcal{L} = \left(\frac{x}{2}, \frac{y}{2}, \frac{1}{2}\right)$. If z has modulus greater than 1, then $\mathcal{L}$ will lie in the upper hemisphere where $\xi > \frac{1}{2}$. If z has modulus less than 1, then $\mathcal{L}$ will lie in the lower hemisphere where $\xi < \frac{1}{2}$. The complex number $z = 0$ corresponds to the *south pole* $\mathcal{S} = (0, 0, 0)$. It is easy to visualize that $z \to \infty$ if and only if $\mathcal{L} \to \mathcal{N}$. Hence $\mathcal{N}$ corresponds to the "ideal" point at infinity. The situation is shown in Figure 2.20.

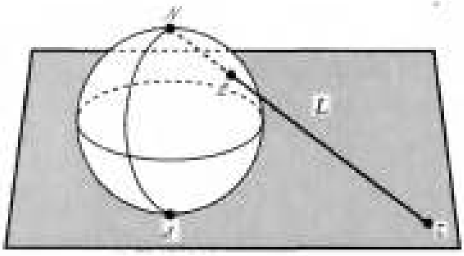

FIGURE 2.20 The Riemann sphere.

Let us reconsider the mapping $w = 1/z$. Let us assign the images $w = \infty$ and $w = 0$ to the points $z = 0$ and $z = \infty$, respectively. The reciprocal transformation can now be written as

$$(5) \quad w = f(z) = \begin{cases} 1/z & \text{when } z \neq 0, z \neq \infty \\ 0 & \text{when } z = \infty \\ \infty & \text{when } z = 0. \end{cases}$$

It is easy to see that the transformation $w = f(z)$ is a one-to-one mapping of the extended complex z plane onto the extended complex w plane. Using property (4) of the point at infinity, it is easy to show that f is a continuous mapping from the extended z plane onto the extended w plane. The details are left for the reader.

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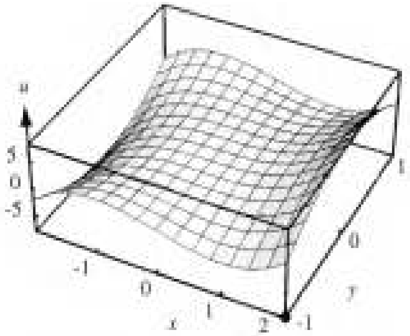

FIGURE 3.2 Graph of $u(x, y) = x^3 - 3xy^2$.

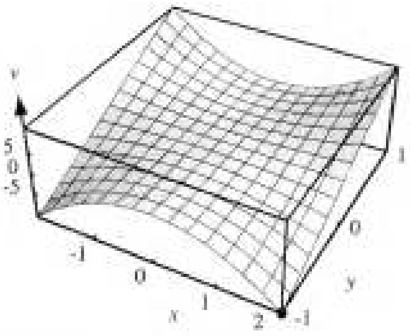

FIGURE 3.3 $v(x, y) = 3x^2y - y^3$.

Complex variable techniques can be used to show that certain combinations of harmonic functions are harmonic. For example, if v is the harmonic conjugate of u, then their product $\phi(x, y) = u(x, y)v(x, y)$ is a harmonic function. This can be verified directly by computing the partial derivatives and showing that equation (2) holds true, but the details are tedious.

If we use complex variable techniques, we can start with the fact that $f(z) = u(x, y) + iv(x, y)$ is an analytic function. Then we observe that the square of f is also an analytic function and is given by $[f(z)]^2 = [u(x, y)]^2 - [v(x, y)]^2 + i2u(x, y)v(x, y)$. Hence the imaginary part of f^2 is $2u(x, y)v(x, y)$ and is a harmonic function. Since a constant multiple of a harmonic function is harmonic, it follows that ϕ is harmonic. It is left as an exercise to show that if u_1 and u_2 are two harmonic functions that are not related in the preceding fashion, then their product need not be harmonic.

Theorem 3.6 (Construction of a Conjugate) *Let $u(x, y)$ be harmonic in an ε-neighborhood of the point (x_0, y_0). Then there exists a conjugate harmonic function $v(x, y)$ defined in this neighborhood, and $f(z) = u(x, y) + iv(x, y)$ is an analytic function.*

Proof The harmonic function u and its conjugate harmonic function v will satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. We can construct $v(x, y)$ in a two-step process. First integrate v_y (which is equal to u_x) with respect to y:

$$(7) \quad v(x, y) = \int u_x(x, y)dy + C(x),$$

where $C(x)$ is a function of x alone (that is, the partial derivative of $C(x)$ with respect to y is zero). Second, we are able to find $C'(x)$ by differentiating equation (7) with respect to x and replacing v_x with $-u_y$ on the left side:

$$(8) \quad -u_y(x, y) = \frac{d}{dx}\int u_x(x, y)dy + C'(x).$$

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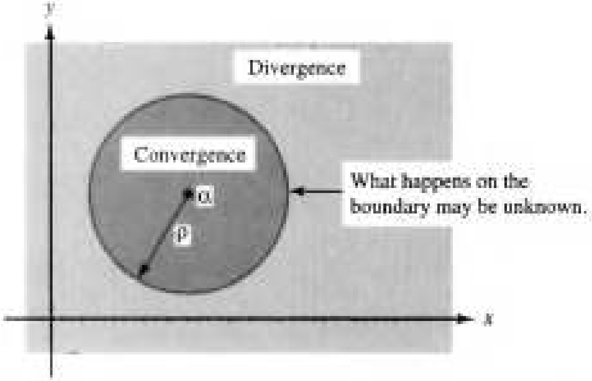
Proof We give a proof assuming $\lim\limits_{n\to\infty} |c_n|^{1/n}$ exists. (A completely general proof making use of the limit supremum can be found in a number of advanced texts.) By Theorem 4.10, the series converges absolutely at those values of z for which $\lim\limits_{n\to\infty} |c_n(z-\alpha)^n|^{1/n} < 1$. This is the same as requiring

$$(1) \quad |z-\alpha| \lim_{n\to\infty} |c_n|^{1/n} < 1.$$

There are three possibilities to consider for the value of $\lim\limits_{n\to\infty} |c_n|^{1/n}$: If the limit equals ∞, inequality (1) holds true iff $z = \alpha$, which puts us in case (i). We will be in case (ii) if $0 < \lim\limits_{n\to\infty} |c_n|^{1/n} < \infty$, since inequality (1) then holds iff $|z-\alpha| < \dfrac{1}{\lim\limits_{n\to\infty} |c_n|^{1/n}}$, i.e., iff $z \in D_\rho(\alpha)$, where $\rho = \dfrac{1}{\lim\limits_{n\to\infty} |c_n|^{1/n}}$. Finally, if the limit equals 0, we will be in case (iii), as the left-hand side of inequality (1) will be 0 for any value of z. Notice we are unable to say for sure what happens with respect to convergence on $C_\rho(\alpha)$. You will see in the exercises that there are various possibilities.

Another way to phrase case (ii) of Theorem 4.11 is to say that the power series $f(z) = \sum\limits_{n=0}^{\infty} c_n(z-\alpha)^n$ converges if $|z-\alpha| < \rho$, and diverges if $|z-\alpha| > \rho$. We call the number ρ the *radius of convergence* of the power series (see Figure 4.2). If we are in case (i) of Theorem 4.11, we say that the radius of convergence is zero, and that the radius of convergence is infinity if we are in case (iii).



What happens on the boundary may be unknown.

FIGURE 4.2 The radius of convergence of a power series.

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the positive fifth root of $e = 2.71828182845904\ldots$. The notation $e^{1/5}$ is ambiguous and might be interpreted as any of the complex fifth roots of the number e that were discussed in Section 1.5:

$$e^{1/5} = 1.22140275816017\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right), \text{ for } k = 0, 1, \ldots, 4.$$

To prevent this confusion, we often use $\exp(z)$ to denote the single-valued exponential function.

Properties of exp(z)

Since identity (3) involves the periodic functions $\cos y$ and $\sin y$, any two points in the z plane that lie on the same vertical line with their imaginary parts differing by an integral multiple of 2π are mapped onto the same point in the w plane. Thus, the complex exponential function is periodic with period $2\pi i$.

EXAMPLE 5.1 The points

$$z_n = \frac{5}{4} + i\left(\frac{11\pi}{6} + 2\pi n\right), \quad \text{for } n = 0, \pm1, \pm2, \ldots$$

in the z plane are mapped onto the single point

$$\begin{aligned} w_0 = \exp(z_n) &= e^{5/4}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}e^{5/4} - \frac{i}{2}e^{5/4} \\ &= 3.022725669\ldots - i1.745171479\ldots \end{aligned}$$

in the w plane, as indicated in Figure 5.1.

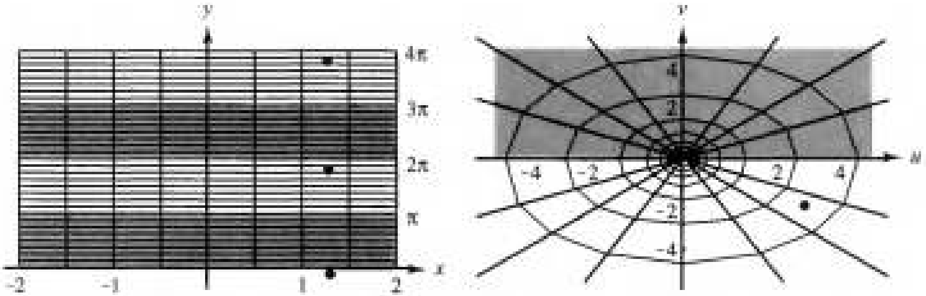
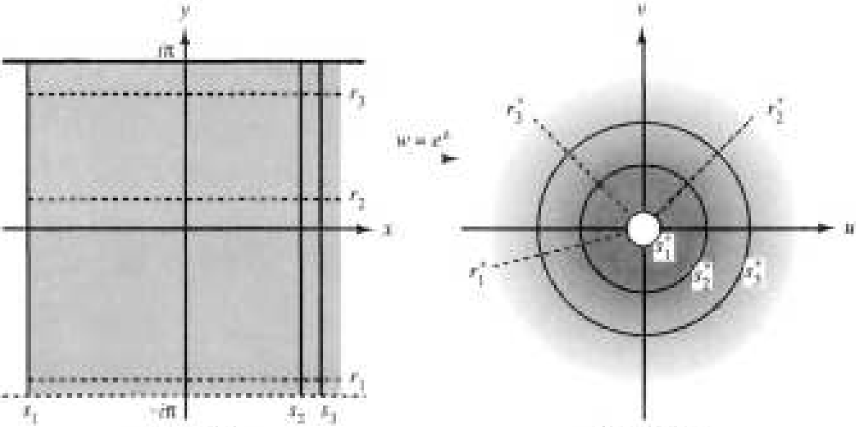


FIGURE 5.1 The points $\{z_n\}$ in the z plane (i.e., the xy plane) and their image $w_0 = \exp(z_n)$ in the w plane (i.e., the uv plane).



The z plane.

The w plane.

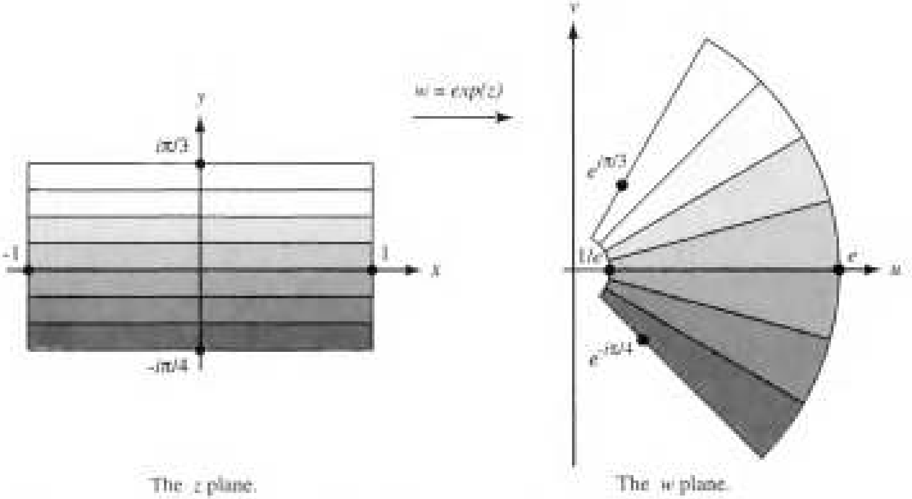
FIGURE 5.2 The fundamental period strip for the mapping $w = \exp(z)$.

EXAMPLE 5.2 Consider a rectangle $R = \{(x, y)\colon a \le x \le b \text{ and } c \le y \le d\}$, where $d - c < 2\pi$. Show that the transformation $w = e^z = e^{x+iy}$ maps R onto a portion of an annular region bounded by two rays.

Solution The image points in the w plane satisfy the following relationships involving the modulus and argument of w:

$$e^a = \left|e^{a+iy}\right| \le \left|e^{x+iy}\right| \le \left|e^{b+iy}\right| = e^b,$$
$$c = \arg(e^{x+ic}) \le \arg(e^{x+iy}) \le \arg(e^{x+id}) \le d,$$

which is a portion of the annulus $\{\rho e^{i\phi}\colon e^a \le \rho \le e^b\}$ in the w plane subtended by the rays $\phi = c$ and $\phi = d$. In Figure 5.3, we show the image of the rectangle $R = \left\{(x, y)\colon -1 \le x \le 1 \text{ and } \frac{-\pi}{4} \le y \le \frac{\pi}{3}\right\}$.



The z plane. The w plane.

FIGURE 5.3 The image of R under the transformation $w = \exp(z)$.

EXERCISES FOR SECTION 5.1

1. Using Definition 5.1, explain why $\exp(0) = e^0 = 1$.
2. Give a careful justification for statements (1) and (2) in the proof of Theorem 5.1, part (iii).
3. An alternate definition for $\exp(z)$: Many texts take the approach of starting with identity (3) as the definition for $f(z) = \exp(z)$. This exercise shows this is a natural approach from the standpoint of differential equations. Our starting point is that we require $f(z)$ to be the solution to an initial-value problem satisfying the following three properties: (1) f is entire, (2) $f'(z) = f(z)$ for all z, and (3) $f(0) = 1$. Suppose $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ satisfies properties (1), (2), and (3).
   (a) Use the result $f'(z) = u_x(x, y) + iv_x(x, y)$ and the requirement $f'(z) = f(z)$ from property (2) to show that $u_x(x, y) - u(x, y) = 0$.
   (b) Show the result in part (a) implies $\frac{\partial}{\partial x}[u(x, y)e^{-x}] = 0$. Note that integrating this last equation partially with respect to x gives $u(x, y)e^{-x} = p(y)$, where $p(y)$ is a function of y alone.
   (c) Using a similar procedure for $v(x, y)$, show we wind up getting the pair of solutions $u(x, y) = p(y)e^x$, and $v(x, y) = q(y)e^x$, where $p(y)$ and $q(y)$ are functions of y alone.
   (d) Now use the Cauchy-Riemann equations to conclude from part (c) that $p(y) = q'(y)$ and $p'(y) = -q(y)$.
   (e) Show the result in part (d) implies $p''(y) + p(y) = 0$ and $q''(y) + q(y) = 0$.
   (f) Given that $f(0) = f(0 + 0i) = u(0, 0) + iv(0, 0) = 1 + 0i$, together with the known solutions to the equations in part (e), conclude that identity (3) follows.
4. Verify properties (4) through (6) given in the text.
5. The questions for this problem relate to Figure 5.2. The shaded portion in the w plane indicates the image of the shaded portion in the z plane, with the darkness of shading indicating the density of corresponding points.

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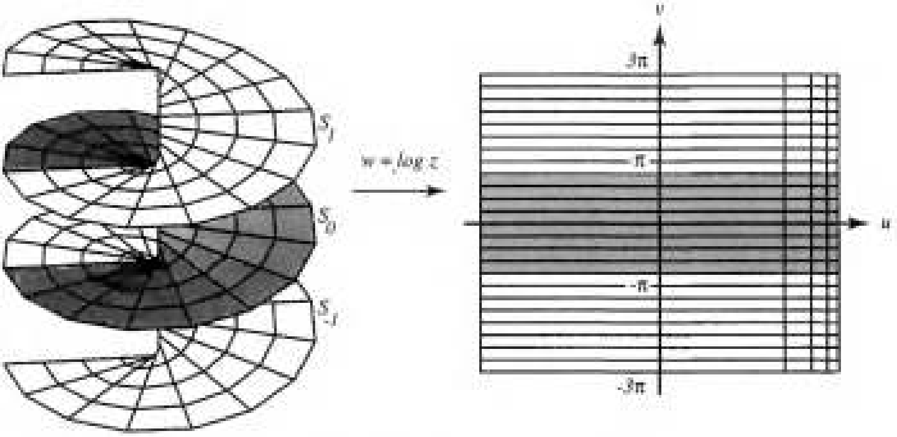

FIGURE 5.7 The Riemann surface for the mapping $w = \log(z)$.

EXERCISES FOR SECTION 5.2

1. Find the principal value $\text{Log}(z) = u + iv$ for the following:
   (a) $\text{Log}(ie^2)$ (b) $\text{Log}(\sqrt{3} - i)$
   (c) $\text{Log}(i\sqrt{2} - \sqrt{2})$ (d) $\text{Log}[(1 + i)^4]$
2. Find *all* the values of $\log(z)$ for the following:
   (a) $\log(-3)$ (b) $\log(4i)$
   (c) $\log 8$ (d) $\log(-\sqrt{3} - i)$
3. Find *all* the values of z for which the following equations hold:
   (a) $\text{Log}(z) = 1 - \dfrac{i\pi}{4}$ (b) $\text{Log}(z - 1) = \dfrac{i\pi}{2}$
   (c) $\exp(z) = -ie$ (d) $\exp(z + 1) = i$
4. Use properties of $\arg(z)$ in Section 1.4 to establish identity (11).
5. Use properties of $\arg(z)$ in Section 1.4 to establish identity (12).
6. Show that $\log(z) = \text{Log}(z) + i2\pi n$ where n is an integer.
7. Let $w = \log(f(z))$. Use implicit differentiation to find dw/dz at points where $f(z)$ is analytic and nonzero.
8. Use implicit differentiation to find dw/dz for the following:
   (a) $w = \log(z^2 - z + 2)$ (b) $w = z \log(z)$
9. Show that $f(z) = \text{Log}(iz)$ is analytic everywhere except at points on the ray given by $y \geq 0$, $x = 0$. Find $f'(z)$.
10. Show that $f(z) = [\text{Log}(z + 5)]/(z^2 + 3z + 2)$ is analytic everywhere except at the points -1, -2 and at points on the ray $x \leq -5$, $y = 0$.
11. Show that if $\text{Re}(z) > 0$, then $\text{Log } z = \frac{1}{2} \ln(x^2 + y^2) + i \arctan(y/x)$, where the principal branch of the real function $\arctan t$ is used; that is, $-\pi/2 < \arctan t < \pi/2$.

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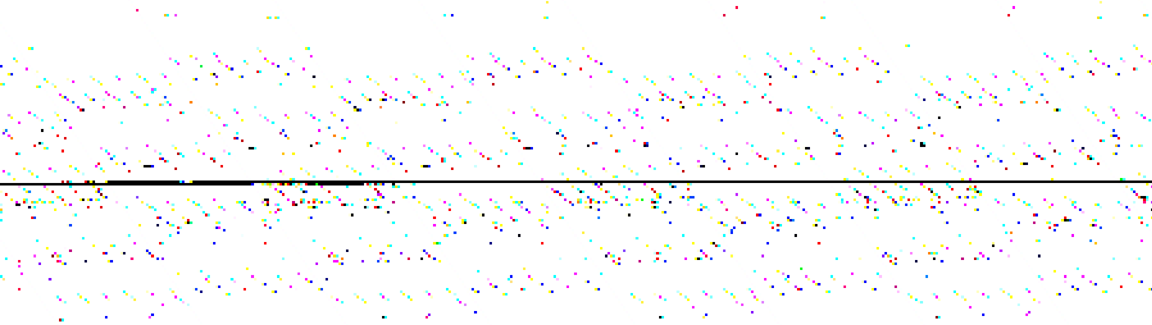

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over contours that are easy to evaluate. We will show how to use the technique of partial fractions together with the Cauchy-Goursat theorem to evaluate certain integrals. In Section 6.4 we will see that the Cauchy-Goursat theorem implies that an analytic function has an antiderivative. To start with, we need to introduce a few new concepts.

We saw in Section 1.6 that with each simple closed contour C there are associated two disjoint domains, each of which has C as its boundary. The contour C divides the plane into two domains. One domain is bounded and is called the *interior* of C, and the other domain is unbounded and is called the *exterior* of C. Figure 6.15 illustrates this concept. This result is known as the Jordan Curve Theorem.

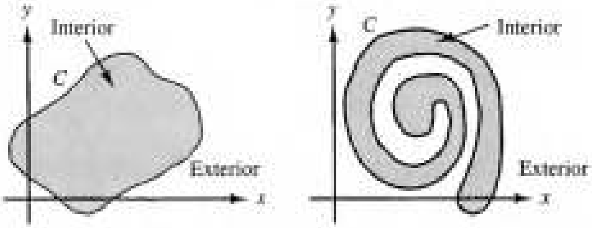
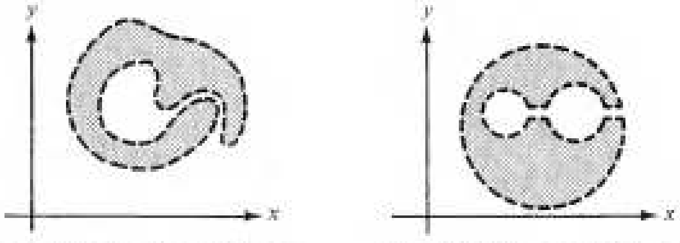
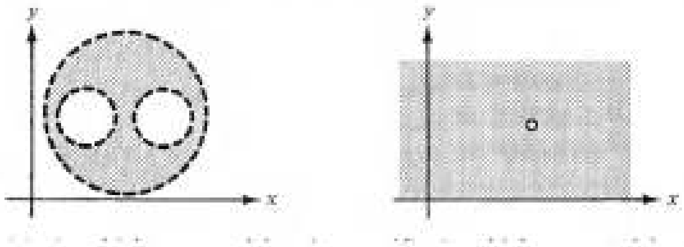


FIGURE 6.15 The interior and exterior of simple closed contours.

In Section 1.6 we saw that a domain D is an open connected set. In particular, if z_1 and z_2 are any pair of points in D, then they can be joined by a curve that lies entirely in D. A domain D is said to be *simply connected* if it has the property that any simple closed contour C contained in D has its interior contained in D. In other words, there are no "holes" in a simply connected domain. A domain that is not simply connected is said to be a *multiply connected domain*. Figure 6.16 illustrates the use of the terms "simply connected" and "multiply connected."



(a) A simply connected domain. (b) A simply connected domain.



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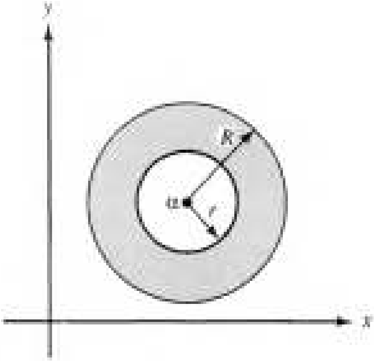

FIGURE 7.3 The closed annulus $\overline{A}(r, R, \alpha)$. The shaded portion is the open annulus $A(r, R, \alpha)$.

Theorem 7.7 *Suppose the Laurent series* $\sum_{n=-\infty}^{\infty} c_n(z-\alpha)^n$ *converges on an annulus* $A(r, R, \alpha)$. *Then the series converges uniformly on any subannulus* $\overline{A}(s, t, \alpha)$, *where* $r < s < t < R$.

Proof According to Definition 7.3,

$$\sum_{n=-\infty}^{\infty} c_n(z-\alpha)^n = \sum_{n=1}^{\infty} c_{-n}(z-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n(z-\alpha)^n.$$

By Theorem 7.2, the series $\sum_{n=0}^{\infty} c_n(z-\alpha)^n$ must converge uniformly on $\overline{D}_t(\alpha)$.

Weierstrass M-test, it is easy to show that the series $\sum_{n=1}^{\infty} c_{-n}(z-\alpha)^{-n}$ con- uniformly on $\{z\colon |z| \geq s\}$ (we leave the details as an exercise). Combining two facts yields the required result.

The main result of this section specifies how functions analytic in an annulus expanded in a Laurent series. In it, we will use symbols of the form $C_\rho^+(\alpha)$, we remind you designate the positively oriented circle with radius ρ and α. That is, $C_\rho^+(\alpha) = \{z\colon |z-\alpha| = \rho\}$, oriented in the positive direction.

Theorem 7.8 (Laurent's Theorem) *Suppose* $0 \leq r < R$, *and that* f *is analytic in the annulus* $A = A(r, R, \alpha) = \{z\colon r < |z-\alpha| < R\}$ *shown in Figure 7.3. If* ρ *is any number such that* $r < \rho < R$, *then for all* $z_0 \in A$, f *has the Laurent series representation*

$$f(z_0) = \sum_{n=-\infty}^{\infty} c_n(z_0-\alpha)^n = \sum_{n=1}^{\infty} c_{-n}(z_0-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n(z_0-\alpha)^n$$

where for $n = 0, 1, 2, \ldots$, *the coefficients* c_{-n} *and* c_n *are given by*

$$(3)\quad c_{-n} = \frac{1}{2\pi i}\int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{-n+1}}\,dz \quad \textit{and} \quad c_n = \frac{1}{2\pi i}\int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}}\,dz.$$

Moreover, the convergence in equation (2) is uniform on any closed subannulus $\overline{A}(s, t, \alpha) = \{z: s \le |z-\alpha| \le t\}$, *where* $r < s < t < R$.

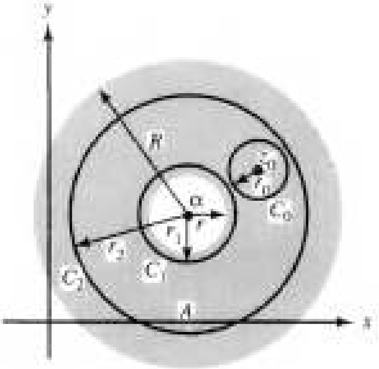

FIGURE 7.4 The annulus A (shaded) and in its interior the circles C_0, C_1, and C_2.

Proof If we can establish equation (2), the uniform convergence on $\overline{A}(s, t, \alpha)$ will follow from Theorem 7.7. Let z_0 be an arbitrary point of A. Choose r_0 small enough so that the circle $C_0 = C_{r_0}^+(z_0)$ is contained in A. Since f is analytic in $D_{r_0}(z_0)$, the Cauchy integral formula gives

$$(4)\quad f(z_0) = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{(z-z_0)}\,dz.$$

Let $C_1 = C_{r_1}^+(\alpha)$ and $C_2 = C_{r_2}^+(\alpha)$, where we choose r_1 and r_2 so that C_0 lies in the region between C_1 and C_2, and $r < r_1 < r_2 < R$, as shown in Figure 7.4. Let D be the domain consisting of the annulus A except for the point z_0. The domain D includes the contours C_0, C_1, and C_2, as well as the region between C_2 and $C_0 + C_1$. In addition, since z_0 does not belong to D, the function $\dfrac{f(z)}{(z-z_0)}$ is analytic on D, so by the extended Cauchy-Goursat theorem we obtain

$$(5)\quad \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)}\,dz = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{(z-z_0)}\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-z_0)}\,dz.$$

Combining equation (5) with equation (4) and rearranging terms gives

$$(6)\quad f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-z_0)}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-z_0)}\,dz.$$

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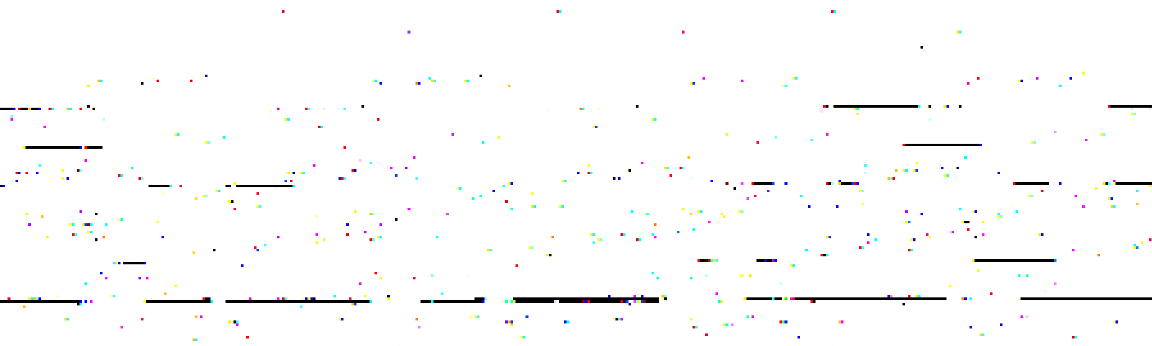

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and $-\pi < c \le \pi$, and the segments $x = a$ and $-\pi < y \le \pi$ form an orthogonal grid in the fundamental period strip. Their images under the mapping $w = \exp z$ are the rays $\rho > 0$ and $\phi = c$ and the circles $|w| = e^a$, respectively. These images form an orthogonal curvilinear grid in the w plane, as shown in Figure 9.9. If $-\pi < c < d \le \pi$, then the rectangle $R = \{x + iy: a < x < b, c < y < d\}$ is mapped one-to-one and onto the region $G = \{\rho e^{i\phi}: e^a < \rho < e^b, c < \phi < d\}$. The inverse mapping is the principal branch of the logarithm $z = \text{Log } w$.

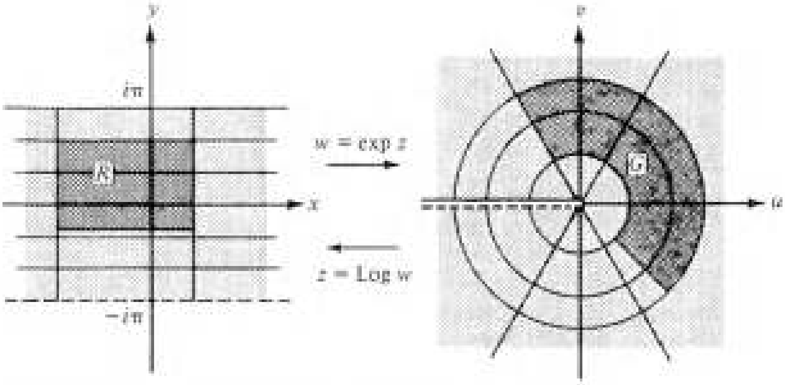


FIGURE 9.9 The conformal mapping $w = \exp z$.

In this section we will show how compositions of conformal transformations are used to construct mappings with specified characteristics.

EXAMPLE 9.8 The transformation $w = f(z) = (e^z - i)/(e^z + i)$ is a one-to-one conformal mapping of the horizontal strip $0 < y < \pi$ onto the disk $|w| < 1$. Furthermore, the x axis is mapped onto the lower semicircle bounding the disk, and the line $y = \pi$ is mapped onto the upper semicircle.

Solution The function $w = f(z)$ can be considered as a composition of the exponential mapping $Z = \exp z$ followed by the bilinear transformation $w = (Z - i)/(Z + i)$. The image of the horizontal strip $0 < y < \pi$ under the mapping $Z = \exp z$ is the upper half plane $\text{Im}(Z) > 0$; the x axis is mapped onto the positive X axis; and the line $y = \pi$ is mapped onto the negative X axis. The bilinear transformation $w = (Z - i)/(Z + i)$ then maps the upper half plane $\text{Im}(Z) > 0$ onto the disk $|w| < 1$; the positive X axis is mapped onto the lower semicircle; and the negative X axis onto the upper semicircle. Figure 9.10 illustrates the composite mapping.

The Riemann Surface for $w = (z^2 - 1)^{1/2}$

Using the other branch of the square root, we find that $w = f_2(z) = -f_1(z)$, is a one-to-one conformal mapping from the domain set D_2 consisting of the z plane slit along the segment $-1 \le x \le 1$, $y = 0$, onto the range set H_2 consisting of the w plane slit along the segment $u = 0$, $-1 \le v \le 1$. The sets D_1 and H_2 for $f_1(z)$ and D_2 and H_2 for $f_2(z)$ are shown in Figure 9.14.

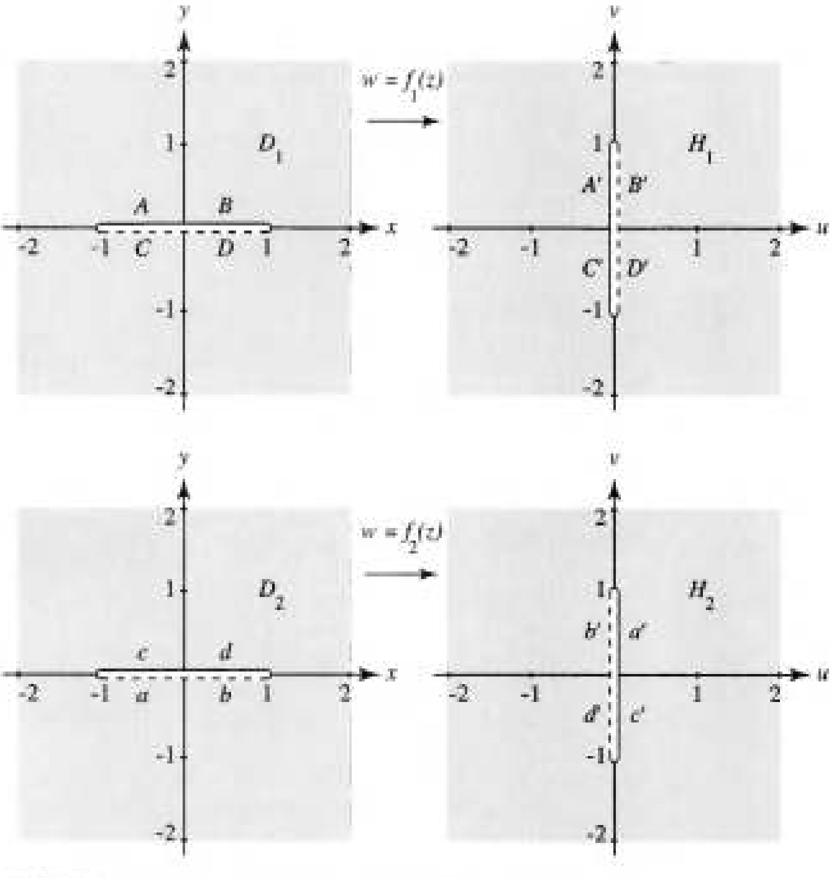
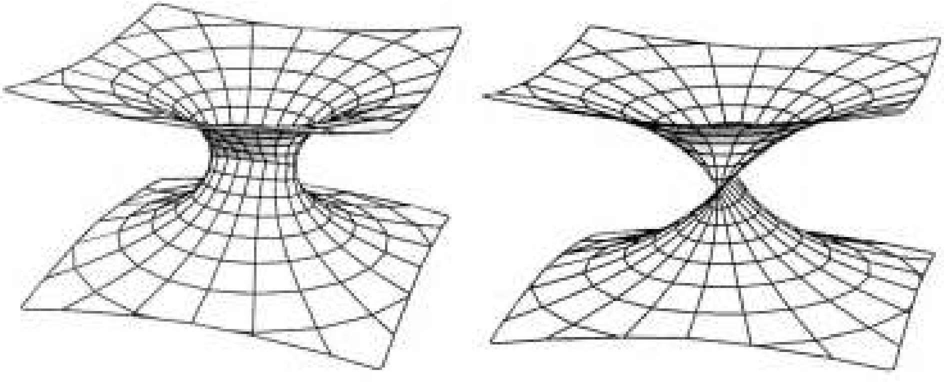


FIGURE 9.14 The mappings $w = f_1(z)$ and $w = f_2(z)$.

The Riemann surface for $w = (z^2 - 1)^{1/2}$ is obtained by gluing the edges of D_1 and D_2 together and H_1 and H_2 together in the following manner. In the domain set, glue edges A to a, B to b, C to c, and D to d. In the image set, glue edges A' to a', B' to b', C' to c', and D' to d'. The result is a Riemann domain surface and Riemann image surface for the mapping, see Figures 9.15(a) and 9.15(b), respectively.



(a) Domain surface D_1 joined to D_2.

(b) Image surface H_1 joined to H_2.

FIGURE 9.15 The Riemann surfaces for the mapping $w = (z^2 - 1)^{1/2}$.

EXERCISES FOR SECTION 9.3

1. Find the image of the semi-infinite strip $0 < x < \pi/2$, $y > 0$, under the transformation $w = \exp iz$.
2. Find the image of the rectangle $0 < x < \ln 2$, $0 < y < \pi/2$, under the transformation $w = \exp z$.
3. Find the image of the first quadrant $x > 0$, $y > 0$, under $w = (2/\pi) \operatorname{Log} z$.
4. Find the image of the annulus $1 < |z| < e$ under $w = \operatorname{Log} z$.
5. Show that the multivalued function $w = \log z$ maps the annulus $1 < |z| < e$ onto the vertical strip $0 < \operatorname{Re}(w) < 1$.
6. Show that $w = (2 - z^2)/z^2$ maps the portion of the right half plane $\operatorname{Re}(z) > 0$ that lies to the right of the hyperbola $x^2 - y^2 = 1$ onto the unit disk $|w| < 1$.
7. Show that the function $w = (e^z - i)/(e^z + i)$ maps the horizontal strip $-\pi < \operatorname{Im}(z) < 0$ onto the region $1 < |w|$.
8. Show that $w = (e^z - 1)/(e^z + 1)$ maps the horizontal strip $|y| < \pi/2$ onto the unit disk $|w| < 1$.
9. Find the image of the upper half plane $\operatorname{Im}(z) > 0$ under $w = \operatorname{Log}[(1 + z)/(1 - z)]$.
10. Find the image of the portion of the upper half plane $\operatorname{Im}(z) > 0$ that lies outside the circle $|z| = 1$ under the transformation $w = \operatorname{Log}[(1 + z)/(1 - z)]$.
11. Show that the function $w = (1 + z)^2/(1 - z)^2$ maps the portion of the disk $|z| < 1$ that lies in the first quadrant onto the portion of the upper half plane $\operatorname{Im}(w) > 0$ that lies outside the unit disk.

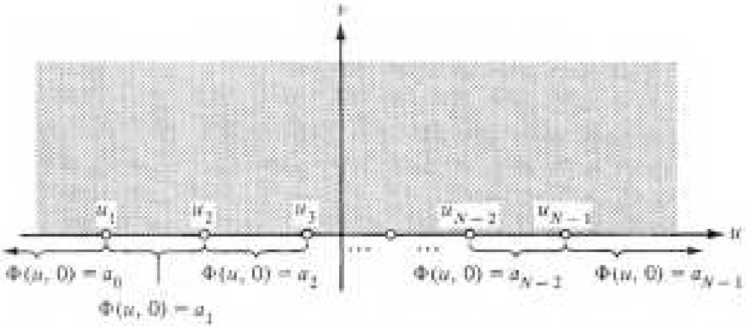


FIGURE 10.4 The boundary conditions for the harmonic function $\Phi(u, v)$ in the statement of Theorem 10.2.

The reader can verify that the boundary conditions are correct for $u < u_1$ and $u > u_{N-1}$, and the result will be established.

EXAMPLE 10.6 Find the function $\phi(x, y)$ that is harmonic in the upper half plane $\mathrm{Re}(z) > 0$, which takes on the boundary values indicated in Figure 10.5.

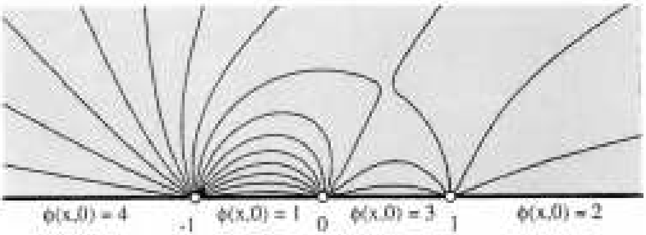


FIGURE 10.5 The boundary values for the Dirichlet problem in Example 10.6.

Solution This is a four-value Dirichlet problem in the upper half plane $\mathrm{Im}(z) > 0$. For the z plane the solution in equation (13) becomes

$$(17) \quad \phi(x, y) = a_3 + \frac{1}{\pi} \sum_{k=1}^{3} (a_{k-1} - a_k) \operatorname{Arg}(z - x_k).$$

Here we have $a_0 = 4$, $a_1 = 1$, $a_2 = 3$, $a_3 = 2$ and $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, which can be substituted into equation (17) to obtain

$$\begin{aligned}\phi(x, y) &= 2 + \frac{4-1}{\pi} \operatorname{Arg}(z+1) + \frac{1-3}{\pi} \operatorname{Arg}(z-0) + \frac{3-2}{\pi} \operatorname{Arg}(z-1) \\ &= 2 + \frac{3}{\pi} \operatorname{Arctan} \frac{y}{x+1} - \frac{2}{\pi} \operatorname{Arctan} \frac{y}{x} + \frac{1}{\pi} \operatorname{Arctan} \frac{y}{x-1}.\end{aligned}$$

EXAMPLE 10.7 Find the function $\phi(x, y)$ that is harmonic in the upper half plane $\mathrm{Re}(z) > 0$, which takes on the boundary values

$$\begin{aligned}\phi(x, 0) &= 1 \quad \text{for } |x| < 1,\\ \phi(x, 0) &= 0 \quad \text{for } |x| > 1.\end{aligned}$$

Solution This is a three-value Dirichlet problem with $a_0 = 0$, $a_1 = 1$, $a_2 = 0$ and $x_1 = -1$ and $x_2 = 1$. Applying formula (13) the solution is

$$\begin{aligned}\phi(x, y) &= 0 + \frac{0-1}{\pi}\operatorname{Arg}(z+1) + \frac{1-0}{\pi}\operatorname{Arg}(z-1)\\ &= -\frac{1}{\pi}\operatorname{Arctan}\frac{y}{x+1} + \frac{1}{\pi}\operatorname{Arctan}\frac{y}{x-1},\end{aligned}$$

and a three-dimensional graph $u = \phi(x, y)$ is shown in Figure 10.6.

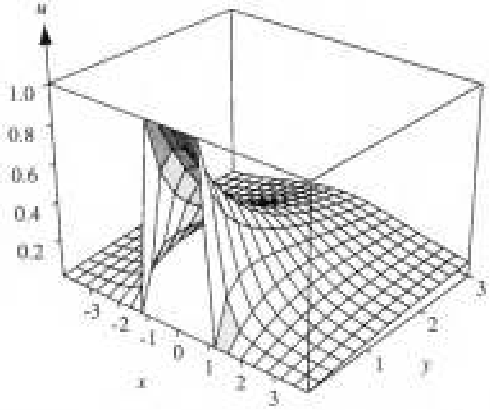


FIGURE 10.6 The graph of $u = \phi(x, y)$ with the boundary values $\phi(x, 0) = 1$ for $|x| < 1$ and $\phi(x, 0) = 0$ for $|x| > 1$.

We now state the *N-value Dirichlet problem* for a *simply connected domain*. Let D be a simply connected domain bounded by the simple closed contour C, and let $z_1, z_2, \ldots, z_N$ denote N points that lie along C in this specified order as C is traversed in the positive (counterclockwise) sense (see Figure 10.7). Let C_k denote the portion of C that lies strictly between z_k and z_{k+1} (for $k = 1, 2, \ldots, N-1$), and let C_N denote the portion that lies strictly between z_N and z_1. Let $a_1, a_2, \ldots, a_N$ be real constants. We want to find a function $\phi(x, y)$ that is harmonic in D and continuous on $D \cup C_1 \cup C_2 \cup \cdots \cup C_N$ that takes on the boundary values:

$$\begin{aligned}(18)\quad \phi(x, y) &= a_1 \quad \text{for } z = x + iy \text{ on } C_1,\\ \phi(x, y) &= a_2 \quad \text{for } z = x + iy \text{ on } C_2,\\ &\ \ \vdots\end{aligned}$$

as shown in Figure 10.9. In this case the method in Example 10.2 can be used to see that $\Phi(u, v)$ is given by

$$(26) \quad \Phi(u, v) = 0 + \frac{1 - 0}{\pi/2} \operatorname{Arg} w = \frac{2}{\pi} \operatorname{Arg} w = \frac{2}{\pi} \operatorname{Arctan} \frac{v}{u}.$$

Using the functions u and v in mapping (22) in equation (26), we find that the solution of the Dirichlet problem in H is

$$\phi(x, y) = \frac{2}{\pi} \operatorname{Arctan} \frac{v(x, y)}{u(x, y)} = \frac{2}{\pi} \operatorname{Arctan} \frac{1 - x^2 - y^2}{2y}.$$

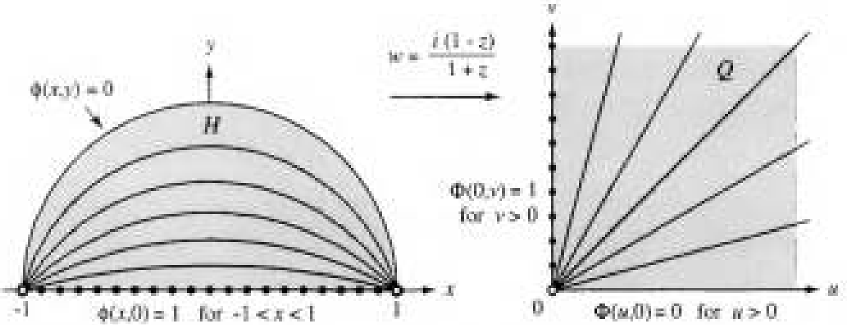


FIGURE 10.9 The Dirichlet problems for the domains H and Q in the solution of Example 10.9.

A three-dimensional graph $u = \phi(x, y)$, in cylindrical coordinates is shown in Figure 10.10.

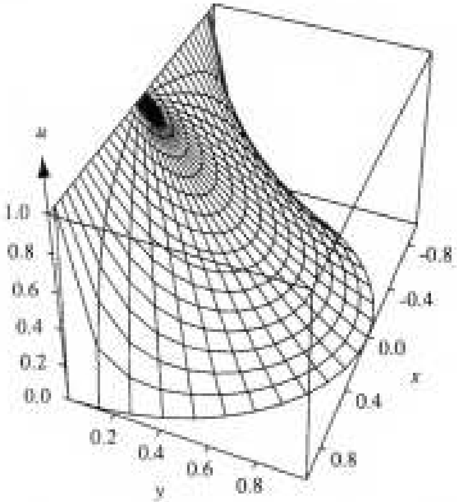


FIGURE 10.10 The graph $u = \frac{2}{\pi} \operatorname{Arctan}\left(\frac{1 - x^2 - y^2}{2y}\right) = \frac{2}{\pi} \operatorname{Arctan}\left(\frac{1 - r^2}{2r \sin\theta}\right)$.

EXAMPLE 10.10 Find a function $\phi(x, y)$ that is harmonic in the quarter disk G: $x > 0$, $y > 0$, $|z| < 1$ and takes on the boundary values

$$\begin{aligned} &\phi(x, y) = 0 \quad \text{for } z = e^{i\theta},\ 0 < \theta < \pi/2, \\ &\phi(x, 0) = 1 \quad \text{for } 0 \le x < 1, \\ &\phi(0, y) = 1 \quad \text{for } 0 \le y < 1. \end{aligned} \tag{27}$$

Solution The function

$$u + iv = z^2 = x^2 - y^2 + i2xy \tag{28}$$

maps the quarter disk onto the upper half-disk H: $v > 0$, $|w| < 1$. The new Dirichlet problem in H is shown in Figure 10.11. From the result of Example 10.9 the solution $\Phi(u, v)$ in H is

$$\Phi(u, v) = \frac{2}{\pi} \operatorname{Arctan} \frac{1 - u^2 - v^2}{2v}. \tag{29}$$

Using equation (28), one can show that $u^2 + v^2 = (x^2 + y^2)^2$ and $2v = 4xy$, which can be used in equation (29) to show that the solution ϕ in G is

$$\phi(x, y) = \frac{2}{\pi} \operatorname{Arctan} \frac{1 - (x^2 + y^2)^2}{4xy}.$$

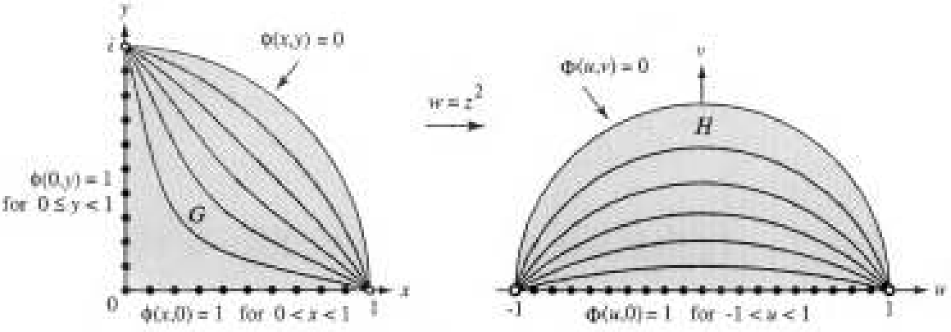


FIGURE 10.11 The Dirichlet problems for the domains G and H in the solution of Example 10.10.

A three-dimensional graph $u = \phi(x, y)$ in cylindrical coordinates is shown in Figure 10.12.

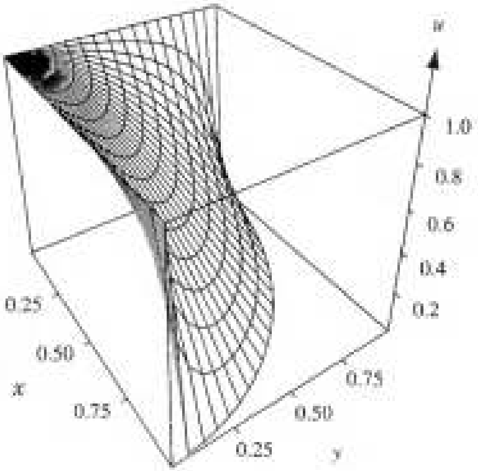


FIGURE 10.12 The graph
$u = \frac{2}{\pi}\operatorname{Arctan}\left[\frac{1-(x^2+y^2)^2}{4xy}\right] = \frac{2}{\pi}\operatorname{Arctan}\left(\frac{1-r^4}{4r^2\cos\theta\sin\theta}\right)$.

EXERCISES FOR SECTION 10.2

For all of the following exercises, find a solution $\phi(x, y)$ of the Dirichlet problem in the domain indicated that takes on the prescribed boundary values.

1. Find the function $\phi(x, y)$ that is harmonic in the horizontal strip $1 < \text{Im}(z) < 2$ and has the boundary values

$$\phi(x, 1) = 6 \quad \text{for all } x, \quad \phi(x, 2) = -3 \quad \text{for all } x.$$

2. Find the function $\phi(x, y)$ that is harmonic in the sector $0 < \text{Arg } z < \pi/3$ and has the boundary values

$$\phi(x, y) = 2 \quad \text{for Arg } z = \pi/3, \quad \phi(x, 0) = 1 \quad \text{for } x > 0.$$

3. Find the function $\phi(x, y)$ that is harmonic in the annulus $1 < |z| < 2$ and has the boundary values

$$\phi(x, y) = 5 \quad \text{when } |z| = 1, \quad \phi(x, y) = 8 \quad \text{when } |z| = 2.$$

4. Find the function $\phi(x, y)$ that is harmonic in the upper half plane $\text{Im}(z) > 0$ and has the boundary values

$$\phi(x, 0) = 1 \quad \text{for } -1 < x < 1, \quad \phi(x, 0) = 0 \quad \text{for } |x| > 1.$$

EXAMPLE 10.12 Find the function $\phi(x, y)$ that is harmonic in the upper half plane $\mathrm{Im}(z) > 0$ and has the boundary values

$$(11) \quad \phi(x, 0) = x \quad \text{for } -1 < x < 1, \quad \phi(x, 0) = 0 \quad \text{for } |x| > 1.$$

Solution Using formula (1), we obtain

$$(12) \quad \phi(x, y) = \frac{y}{\pi}\int_{-1}^{1} \frac{t\,dt}{(x-t)^2 + y^2}$$
$$= \frac{y}{\pi}\int_{-1}^{1} \frac{(x-t)(-1)\,dt}{(x-t)^2 + y^2} + \frac{x}{\pi}\int_{-1}^{1} \frac{y\,dt}{(x-t)^2 + y^2}.$$

Using techniques of calculus and equations (7), we find that the solution in equation (12) is

$$\phi(x, y) = \frac{y}{2\pi}\ln\frac{(x-1)^2 + y^2}{(x+1)^2 + y^2} + \frac{x}{\pi}\operatorname{Arctan}\frac{y}{x-1} - \frac{x}{\pi}\operatorname{Arctan}\frac{y}{x+1}.$$

The function $\phi(x, y)$ is continuous in the upper half plane and on the boundary $\phi(x, 0)$ has discontinuities at $x = \pm 1$ on the real axis. The graph in Figure 10.14 shows this phenomenon.

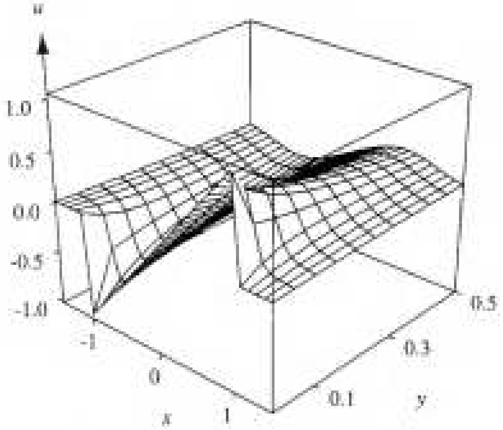


FIGURE 10.14 The graph of $u = \phi(x, y)$ with the boundary values $\phi(x, 0) = x$ for $|x| < 1$ and $\phi(x, 0) = 0$ for $|x| > 1$.

EXAMPLE 10.13 Find $\phi(x, y)$, harmonic in the upper half plane $\mathrm{Im}(z) > 0$, that has the boundary values $\phi(x, 0) = x$ for $|x| < 1$, $\phi(x, 0) = -1$ for $x < -1$, and $\phi(x, 0) = 1$ for $x > 1$.

Solution Using techniques from Section 10.2, the function

$$v(x, y) = 1 - \frac{1}{\pi}\operatorname{Arctan}\frac{y}{x+1} - \frac{1}{\pi}\operatorname{Arctan}\frac{y}{x-1}$$

[illegible]

[illegible]

[illegible]

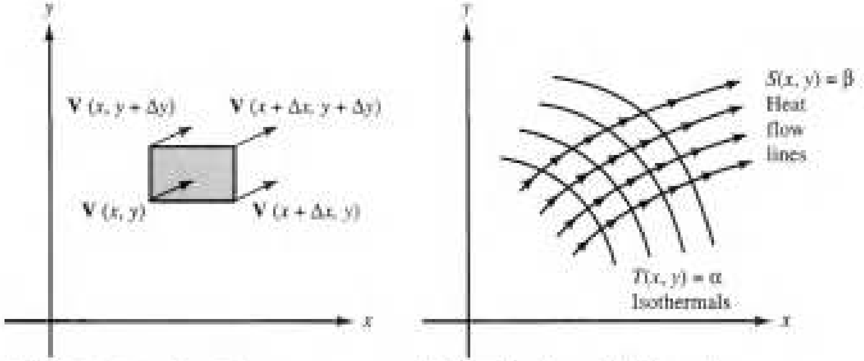
[illegible]

[illegible]

[illegible]

and the amount of heat flowing out of the left edge is

(4) $\mathbf{V} \cdot \mathbf{N}_2 \, \Delta s_2 = -K[T_x(x, y) + iT_y(x, y)] \cdot (-1 + 0i) \, \Delta y = KT_x(x, y) \, \Delta y.$



(a) The direction of heat flow. (b) Heat flow lines and isothermals.

FIGURE 10.16 Steady state temperatures.

If we add the contributions in equations (3) and (4), the result is

(5) $$-K\left[\frac{T_x(x + \Delta x, y) - T_x(x, y)}{\Delta x}\right] \Delta x \, \Delta y \approx -KT_{xx}(x, y) \, \Delta x \, \Delta y.$$

In a similar fashion it is found that the contribution for the amount of heat flowing out of the top and bottom edges is

(6) $$-K\left[\frac{T_y(x, y + \Delta y) - T_y(x, y)}{\Delta y}\right] \Delta x \, \Delta y \approx -KT_{yy}(x, y) \, \Delta x \, \Delta y.$$

Adding the quantities in equations (5) and (6), we find that the net heat flowing out of the rectangle is approximated by the equation

(7) $-K[T_{xx}(x, y) + T_{yy}(x, y)] \, \Delta x \, \Delta y = 0,$

which implies that $T(x, y)$ satisfies Laplace's equation and is a harmonic function.

If the domain in which $T(x, y)$ is defined is simply connected, then a conjugate harmonic function $S(x, y)$ exists, and

(8) $F(z) = T(x, y) + iS(x, y)$

is an analytic function. The curves $T(x, y) = K_1$ are called *isothermals* and are lines connecting points of the same temperature. The curves $S(x, y) = K_2$ are called the *heat flow lines*, and one can visualize the heat flowing along these curves from points of higher temperature to points of lower temperature. The situation is illustrated in Figure 10.16(b).

Boundary value problems for steady state temperatures are realizations of the Dirichlet problem where the value of the harmonic function $T(x, y)$ is interpreted as the temperature at the point (x, y).

EXAMPLE 10.14 Suppose that two parallel planes are perpendicular to the z plane and pass through the horizontal lines $y = a$ and $y = b$ and that the temperature is held constant at the values $T(x, a) = T_1$ and $T(x, b) = T_2$, respectively, on these planes. Then T is given by

$$(9) \quad T(x, y) = T_1 + \frac{T_2 - T_1}{b - a}(y - a).$$

Solution It is reasonable to make the assumption that the temperature at all points on the plane passing through the line $y = y_0$ is constant. Hence $T(x, y) = t(y)$, where $t(y)$ is a function of y alone. Laplace's equation implies that $t''(y) = 0$, and an argument similar to that in Example 10.1 will show that the solution $T(x, y)$ has the form given in equation (9).

The isothermals $T(x, y) = \alpha$ are easily seen to be horizontal lines. The conjugate harmonic function is

$$S(x, y) = \frac{T_1 - T_2}{b - a}x,$$

and the heat flow lines $S(x, y) = \beta$ are vertical segments between the horizontal lines. If $T_1 > T_2$, then the heat flows along these segments from the plane through $y = a$ to the plane through $y = b$ as illustrated in Figure 10.17.

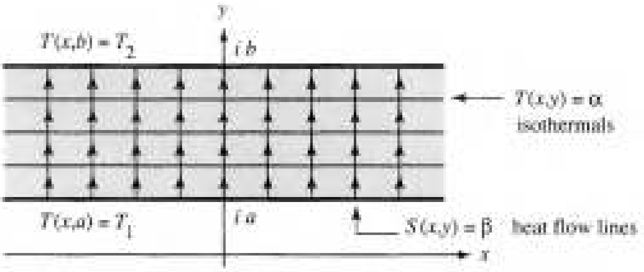


FIGURE 10.17 The temperature between parallel planes where $T_1 > T_2$.

EXAMPLE 10.15 Find the temperature $T(x, y)$ at each point in the upper half plane $\text{Im}(z) > 0$ if the temperature along the x axis satisfies

$$(10) \quad T(x, 0) = T_1 \quad \text{for } x > 0 \quad \text{and} \quad T(x, 0) = T_2 \quad \text{for } x < 0.$$

Solution Since $T(x, y)$ is a harmonic function, this is an example of a Dirichlet problem. From Example 10.2 it follows that the solution is

$$(11) \quad T(x, y) = T_1 + \frac{T_2 - T_1}{\pi}\text{Arg } z.$$

The isothermals $T(x, y) = \alpha$ are rays emanating from the origin. The conjugate harmonic function is $S(x, y) = (1/\pi)(T_1 - T_2) \ln|z|$, and the heat flow lines $S(x, y) = \beta$ are semicircles centered at the origin. If $T_1 > T_2$, then the heat flows counterclockwise along the semicircles as shown in Figure 10.18.

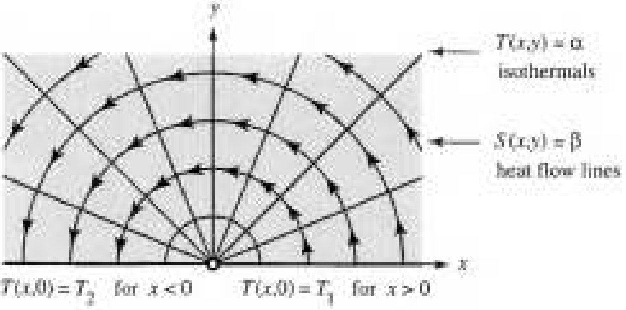


FIGURE 10.18 The temperature $T(x, y)$ in the upper half plane where $T_1 > T_2$.

EXAMPLE 10.16 Find the temperature $T(x, y)$ at each point in the upper half-disk H: $\text{Im}(z) > 0$, $|z| < 1$ if the temperature at points on the boundary satisfies

$$(12) \quad \begin{aligned} &T(x, y) = 100 \quad \text{for } z = e^{i\theta},\ 0 < \theta < \pi, \\ &T(x, 0) = 50 \quad \text{for } -1 < x < 1. \end{aligned}$$

Solution As discussed in Example 10.9, the function

$$(13) \quad u + iv = \frac{i(1 - z)}{1 + z} = \frac{2y}{(x + 1)^2 + y^2} + i\frac{1 - x^2 - y^2}{(x + 1)^2 + y^2}$$

is a one-to-one conformal mapping of the half-disk H onto the first quadrant Q: $u > 0$, $v > 0$. The conformal map (13) gives rise to a new problem of finding the temperature $T^*(u, v)$ that satisfies the boundary conditions

$$(14) \quad T^*(u, 0) = 100 \quad \text{for } u > 0 \quad \text{and} \quad T^*(0, v) = 50 \quad \text{for } v > 0.$$

If we use Example 10.2, the harmonic function $T^*(u, v)$ is given by

$$(15) \quad T^*(u, v) = 100 + \frac{50 - 100}{\pi/2} \text{Arg } w = 100 - \frac{100}{\pi} \text{Arctan } \frac{v}{u}.$$

Substituting the expressions for u and v in mapping (13) into equation (15) yields the desired solution

$$T(x, y) = 100 - \frac{100}{\pi} \operatorname{Arctan} \frac{1 - x^2 - y^2}{2y}.$$

The isothermals $T(x, y) =$ constant are circles that pass through the points ±1 as shown in Figure 10.19.

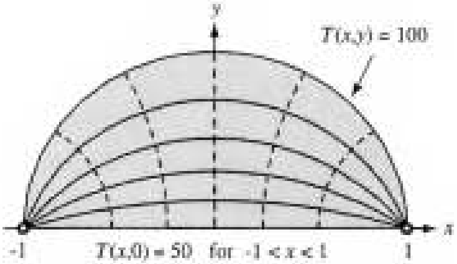


FIGURE 10.19 The temperature $T(x, y)$ in a half-disk.

We now turn our attention to the problem of finding the steady state temperature function $T(x, y)$ inside the simply connected domain D whose boundary consists of three adjacent curves C_1, C_2, and C_3, where $T(x, y) = T_1$ along C_1, $T(x, y) = T_2$ along C_2, and the region is insulated along C_3. Zero heat flowing across C_3 implies that

$$\text{(16)} \quad \mathbf{V}(x, y) \cdot \mathbf{N}(x, y) = -K\mathbf{N}(x, y) \cdot \operatorname{grad} T(x, y) = 0,$$

where $\mathbf{N}(x, y)$ is perpendicular to C_3. This means that the direction of heat flow must be parallel to this portion of the boundary. In other words, C_3 must be part of a heat flow line $S(x, y) =$ constant and the isothermals $T(x, y) =$ constant intersect C_3 orthogonally.

This problem can be solved if we can find a conformal mapping

$$\text{(17)} \quad w = f(z) = u(x, y) + iv(x, y)$$

from D onto the semi-infinite strip G: $0 < u < 1$, $v > 0$ so that the image of the curve C_1 is the ray $u = 0$, $v > 0$; the image of the curve C_2 is the ray given by $u = 1$, $v > 0$; and the thermally insulated curve C_3 is mapped onto the thermally insulated segment $0 < u < 1$ of the u axis, as shown in Figure 10.20.

EXERCISES FOR SECTION 10.5

1. Show that $H(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies Laplace's equation $H_{xx} + H_{yy} + H_{zz} = 0$ in three-dimensional Cartesian space but that $h(x, y) = 1/\sqrt{x^2 + y^2}$ *does not* satisfy equation $h_{xx} + h_{yy} = 0$ in two-dimensional Cartesian space.
2. Find the temperature function $T(x, y)$ in the infinite strip bounded by the lines $y = -x$ and $y = 1 - x$ that satisfies the boundary values in Figure 10.22.

$T(x, -x) = 25$ for all x,
$T(x, 1 - x) = 75$ for all x.

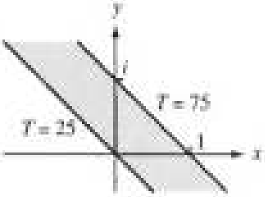


FIGURE 10.22 Accompanies Exercise 2.

3. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary values in Figure 10.23. *Hint*: Use $w = z^2$.

$T(x, 0) = 10$ for $x > 1$,
$T(x, 0) = 20$ for $0 < x < 1$,
$T(0, y) = 20$ for $0 \leq y < 1$,
$T(0, y) = 10$ for $y > 1$.

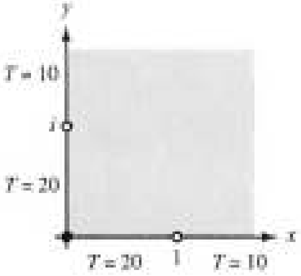


FIGURE 10.23 Accompanies Exercise 3.

4. Find the temperature function $T(x, y)$ inside the unit disk $|z| < 1$ that satisfies the boundary values in Figure 10.24. *Hint*: Use $w = i(1 - z)/(1 + z)$.

$T(x, y) = 20$ for $z = e^{i\theta}$, $0 < \theta < \dfrac{\pi}{2}$,

$T(x, y) = 60$ for $z = e^{i\theta}$, $\dfrac{\pi}{2} < \theta < 2\pi$.

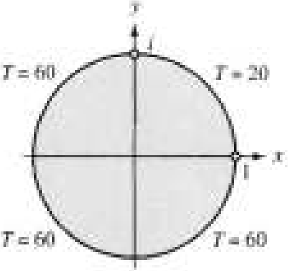


FIGURE 10.24 Accompanies Exercise 4.

5. Find the temperature function $T(x, y)$ in the semi-infinite strip $-\pi/2 < x < \pi/2$, $y > 0$ that satisfies the boundary values in Figure 10.25. *Hint*: Use $w = \sin z$.

$$T\left(\frac{\pi}{2}, y\right) = 100 \quad \text{for } y > 0,$$

$$T(x, y) = 0 \qquad \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2},$$

$$T\left(\frac{-\pi}{2}, y\right) = 100 \quad \text{for } y > 0.$$

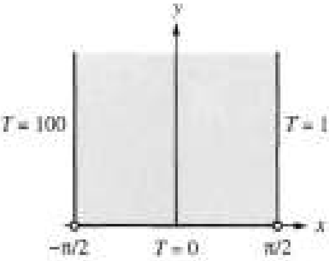


FIGURE 10.25 Accompanies Exercise 5.

6. Find the temperature function $T(x, y)$ in the domain $r > 1$, $0 < \theta < \pi$ that satisfies the boundary values in Figure 10.26. *Hint*: $w = i(1 - z)/(1 + z)$.

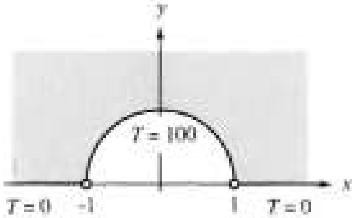


FIGURE 10.26 Accompanies Exercise 6.

$T(x, 0) = 0 \quad \text{for } x > 1,$
$T(x, 0) = 0 \quad \text{for } x < -1,$
$T(x, y) = 100 \quad \text{if } z = e^{i\theta},\ 0 < \theta < \pi.$

7. Find the temperature function $T(x, y)$ in the domain $1 < r < 2$, $0 < \theta < \pi/2$ that satisfies the boundary conditions in Figure 10.27. *Hint*: Use $w = \text{Log } z$.

$$\begin{aligned} T(x, y) &= 0 && \text{for } r = e^{i\theta}, 0 < \theta < \frac{\pi}{2}, \\ T(x, y) &= 50 && \text{for } r = 2e^{i\theta}, 0 < \theta < \frac{\pi}{2}, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } 1 < x < 2, \\ \frac{\partial T}{\partial n} = T_x(0, y) &= 0 && \text{for } 1 < y < 2. \end{aligned}$$

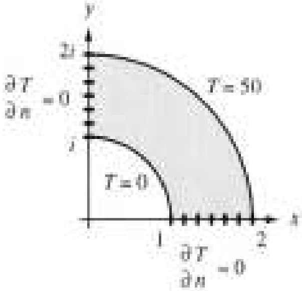

y
2i
$\frac{\partial T}{\partial n} = 0$
T = 50
i
T = 0
1
2
x
$\frac{\partial T}{\partial n} = 0$

FIGURE 10.27 Accompanies Exercise 7.

8. Find the temperature function $T(x, y)$ in the domain $0 < r < 1$, $0 < \text{Arg } z < \alpha$ that satisfies the boundary conditions in Figure 10.28. *Hint*: Use $w = \text{Log } z$.

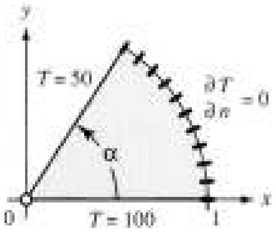


FIGURE 10.28 Accompanies Exercise 8.

$$\begin{aligned} T(x, 0) &= 100 && \text{for } 0 < x < 1, \\ T(x, y) &= 50 && \text{for } z = re^{i\alpha}, 0 < r < 1, \\ \frac{\partial T}{\partial n} &= 0 && \text{for } z = e^{i\theta}, 0 < \theta < \alpha. \end{aligned}$$

9. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary conditions in Figure 10.29. *Hint:* Use $w = \text{Arcsin } z^2$.

$$\begin{aligned} T(x, 0) &= 100 && \text{for } x > 1, \\ T(0, y) &= -50 && \text{for } y > 1, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } 0 < x < 1, \\ \frac{\partial T}{\partial n} = T_x(0, y) &= 0 && \text{for } 0 < y < 1. \end{aligned}$$

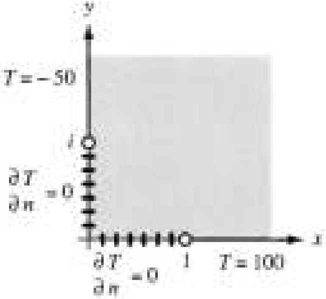

y
T = − 50
i
$\frac{\partial T}{\partial n} = 0$
x
$\frac{\partial T}{\partial n} = 0$
1
T = 100

FIGURE 10.29 Accompanies Exercise 9.

10. Find the temperature function $T(x, y)$ in the infinite strip $0 < y < \pi$ that satisfies the boundary conditions in Figure 10.30. *Hint:* Use $w = e^z$.

$$\begin{aligned} T(x, 0) &= 50 && \text{for } x > 0, \\ T(x, \pi) &= -50 && \text{for } x > 0, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } x < 0, \\ \frac{\partial T}{\partial n} = T_y(x, \pi) &= 0 && \text{for } x < 0. \end{aligned}$$

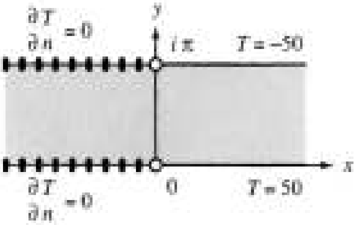

$\frac{\partial T}{\partial n} = 0$
y
iπ
T = −50
x
$\frac{\partial T}{\partial n} = 0$
0
T = 50

FIGURE 10.30 Accompanies Exercise 10.

11. Find the temperature function $T(x, y)$ in the upper half plane $\text{Im}(z) > 0$ that satisfies the boundary conditions in Figure 10.31. *Hint:* Use $w = 1/z$.

$$\begin{aligned} T(x, 0) &= 100 && \text{for } 0 < x < 1, \\ T(x, 0) &= -100 && \text{for } -1 < x < 0, \\ \frac{\partial T}{\partial n} &= T_y(x, 0) = 0 && \text{for } x > 1, \\ \frac{\partial T}{\partial n} &= T_y(x, 0) = 0 && \text{for } x < -1. \end{aligned}$$

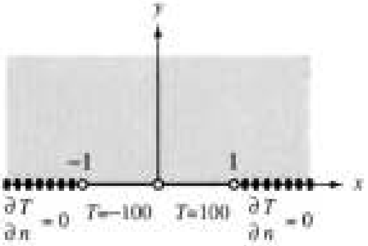

y
−1
1
x
$\frac{\partial T}{\partial n} = 0$
T=−100
T=100
$\frac{\partial T}{\partial n} = 0$

FIGURE 10.31 Accompanies Exercise 11.

12. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary conditions in Figure 10.32.

$$\begin{aligned} T(x, 0) &= 50 && \text{for } x > 0, \\ T(0, y) &= -50 && \text{for } y > 1, \\ \frac{\partial T}{\partial n} &= T_x(0, y) = 0 && \text{for } 0 < y < 1. \end{aligned}$$

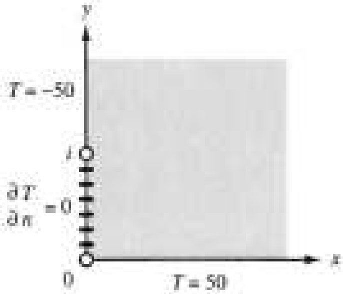


FIGURE 10.32 Accompanies Exercise 12.

13. For the temperature function

$$T(x, y) = 100 - \frac{100}{\pi} \arctan \frac{1 - x^2 - y^2}{2y}$$

in the upper half-disk $|z| < 1$, $\mathrm{Im}(z) > 0$, show that the isothermals $T(x, y) = \alpha$ are portions of circles that pass through the points $+1$ and -1 as illustrated in Figure 10.33.

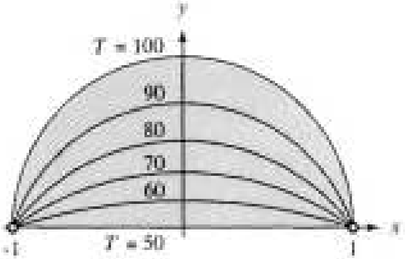


FIGURE 10.33 Accompanies Exercise 13.

14. For the temperature function

$$T(x, y) = \frac{300}{\pi}\,\mathrm{Re}(\mathrm{Arcsin}\ z)$$

in the upper half plane $\mathrm{Im}(z) > 0$, show that the isothermals $T(x, y) = \alpha$ are portions of hyperbolas that have foci at the points ± 1 as illustrated in Figure 10.34.

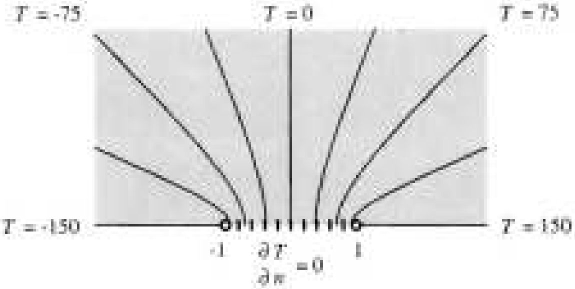


FIGURE 10.34 Accompanies Exercise 14.

15. Find the temperature function in the portion of the upper half plane $\mathrm{Im}(z) > 0$ that lies inside the ellipse

$$\frac{x^2}{\cosh^2 2} + \frac{y^2}{\sinh^2 2} = 1$$

and satisfies the boundary conditions given in Figure 10.35. *Hint:* Use $w = \mathrm{Arcsin}\ z$.

$$\begin{aligned} T(x, y) &= 80 && \text{for } (x, y) \text{ on the ellipse,}\\ T(x, 0) &= 40 && \text{for } -1 < x < 1,\\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{when } 1 < |x| < \cosh 2. \end{aligned}$$

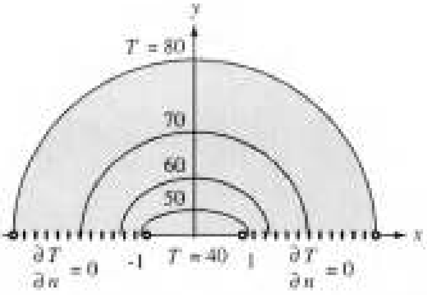

y
T = 80
70
60
50
x
$\frac{\partial T}{\partial n} = 0$
-1
T = 40
1
$\frac{\partial T}{\partial n} = 0$

FIGURE 10.35 Accompanies Exercise 15.

10.6 Two-Dimensional Electrostatics

A two-dimensional electrostatic field is produced by a system of charged wires, plates, and cylindrical conductors that are perpendicular to the z plane. The wires, plates, and cylinders are assumed to be so long that the effects at the ends can be neglected as mentioned in Section 10.4. This sets up an electric field $\mathbf{E}(x, y)$ that can be interpreted as the force acting on a unit positive charge placed at the point (x, y). In the study of electrostatics the vector field $\mathbf{E}(x, y)$ is shown to be *conservative* and is derivable from a function $\phi(x, y)$, called the *electrostatic potential*, as expressed by the equation

(1) $\mathbf{E}(x, y) = -\text{grad}\,\phi(x, y) = -\phi_x(x, y) - i\phi_y(x, y).$

If we make the additional assumption that there are no charges within the domain D, then Gauss' law for electrostatic fields implies that the line integral of the outward normal component of $\mathbf{E}(x, y)$ taken around any small rectangle lying inside D is identically zero. A heuristic argument similar to the one for steady state temperatures with $T(x, y)$ replaced by $\phi(x, y)$ will show that the value of the line integral is

(2) $-[\phi_{xx}(x, y) + \phi_{yy}(x, y)]\,\Delta x\,\Delta y.$

Since the quantity in expression (2) is zero, we conclude that $\phi(x, y)$ is a harmonic function. We let $\psi(x, y)$ denote the harmonic conjugate, and

(3) $F(z) = \phi(x, y) + i\psi(x, y)$

is the complex potential (not to be confused with the electrostatic potential).

The curves $\phi(x, y) = K_1$ are called the *equipotential curves*, and the curves $\psi(x, y) = K_2$ are called the *lines of flux*. If a small test charge is allowed to move under the influence of the field $\mathbf{E}(x, y)$, then it will travel along a line of flux. Boundary value problems for the potential function $\phi(x, y)$ are mathematically the same as those for steady state heat flow, and they are realizations of the Dirichlet problem where the harmonic function is $\phi(x, y)$.

[illegible]

[illegible]

[illegible]

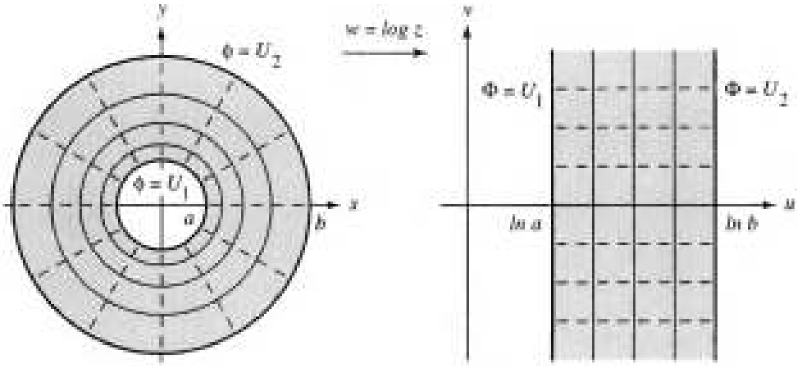


FIGURE 10.36 The electric field in a coaxial cylinder where $U_2 < U_1$.

Using the result of Example 10.1, we see that $\Phi(u, v)$ is

$$(9) \quad \Phi(u, v) = \frac{600}{\pi} u.$$

As in the discussion of Example 10.17, the solution in the z plane is

$$(10) \quad \phi(x, y) = \frac{600}{\pi} \operatorname{Re}(\operatorname{Arcsin} z)$$
$$= \frac{600}{\pi} \operatorname{Arcsin}\left[\frac{\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}}{2}\right].$$

Several equipotential curves are shown in Figure 10.37.

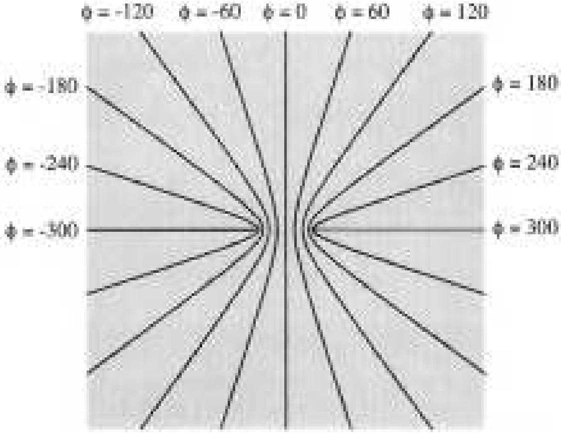


FIGURE 10.37 The electric field produced by two charged half planes that are perpendicular to the complex plane.

EXAMPLE 10.21 Find the electrical potential $\phi(x, y)$ in the disk D: $|z| < 1$ that satisfies the boundary values

$$(11)\quad \phi(x, y) = 80 \quad \text{for } z \text{ on } C_1 = \left\{z = e^{i\theta}: 0 < \theta < \frac{\pi}{2}\right\} \quad \text{and}$$

$$\phi(x, y) = 0 \quad \text{for } z \text{ on } C_2 = \left\{z = e^{i\theta}: \frac{\pi}{2} < \theta < 2\pi\right\}.$$

Solution The mapping $w = S(z) = [(1 - i)(z - i)]/(z - 1)$ is a one-to-one conformal mapping of D onto the upper half plane $\text{Im}(w) > 0$ with the property that C_1 is mapped onto the negative u axis and C_2 is mapped onto the positive u axis. The potential $\Phi(u, v)$ in the upper half plane that satisfies the new boundary values

$$(12)\quad \Phi(u, 0) = 80 \quad \text{for } u < 0 \quad \text{and} \quad \Phi(u, 0) = 0 \quad \text{for } u > 0$$

is given by

$$(13)\quad \Phi(u, v) = \frac{80}{\pi}\,\text{Arg}\, w = \frac{80}{\pi}\,\text{Arctan}\,\frac{v}{u}.$$

A straightforward calculation shows that

$$(14)\quad u + iv = S(z) = \frac{(x - 1)^2 + (y - 1)^2 - 1 + i(1 - x^2 - y^2)}{(x - 1)^2 + y^2}.$$

The functions u and v in equation (14) can be substituted into equation (13) to obtain

$$\phi(x, y) = \frac{80}{\pi}\,\text{Arctan}\,\frac{1 - x^2 - y^2}{(x - 1)^2 + (y - 1)^2 - 1}.$$

The level curve $\Phi(u, v) = \alpha$ in the upper half plane is a ray emanating from the origin, and the preimage $\phi(x, y) = \alpha$ in the unit disk is an arc of a circle that passes through the points 1 and i. Several level curves are illustrated in Figure 10.38.

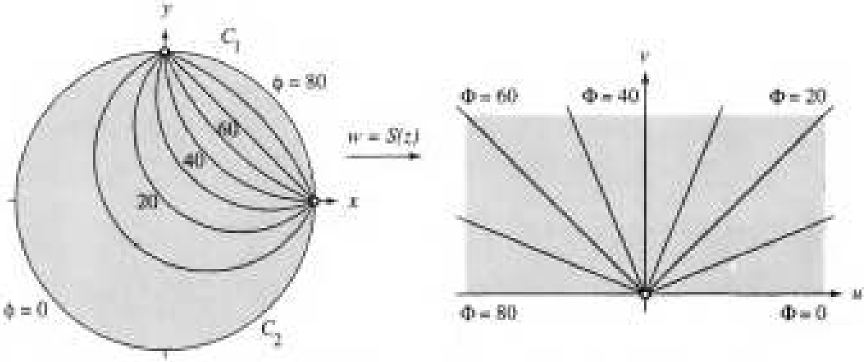


FIGURE 10.38 The potentials ϕ and Φ that are discussed in Example 10.21.

EXERCISES FOR SECTION 10.6

1. Find the electrostatic potential $\phi(x, y)$ between the two coaxial cylinders $r = 1$ and $r = 2$ that has the boundary values as shown in Figure 10.39.

$\phi(x, y) = 100$ when $|z| = 1$,
$\phi(x, y) = 200$ when $|z| = 2$.

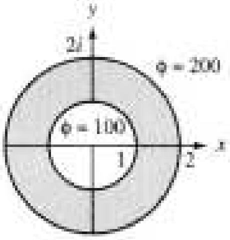


FIGURE 10.39 Accompanies Exercise 1.

2. Find the electrostatic potential $\phi(x, y)$ in the upper half plane $\text{Im}(z) > 0$ that satisfies the boundary values as shown in Figure 10.40.

$\phi(x, 0) = 100$ for $x > 1$
$\phi(x, 0) = 0$ for $-1 < x < 1$
$\phi(x, 0) = -100$ for $x < -1$

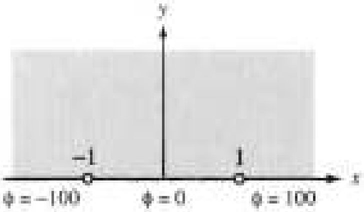


FIGURE 10.40 Accompanies Exercise 2.

3. Find the electrostatic potential $\phi(x, y)$ in the crescent-shaped region that lies inside the disk $|z - 2| < 2$ and outside the circle $|z - 1| = 1$ that satisfies the boundary values as shown in Figure 10.41. *Hint*: Use $w = 1/z$.

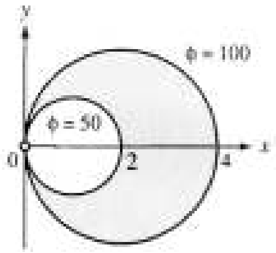


FIGURE 10.41 Accompanies Exercise 3.

$$\phi(x, y) = 100 \quad \text{for } |z - 2| = 2,\ z \neq 0,$$
$$\phi(x, y) = 50 \quad \text{for } |z - 1| = 1,\ z \neq 0.$$

4. Find the electrostatic potential $\phi(x, y)$ in the semi-infinite strip $-\pi/2 < x < \pi/2$, $y > 0$ that has the boundary values as shown in Figure 10.42.

$$\phi\left(\frac{\pi}{2}, y\right) = 0 \quad \text{for } y > 0,$$
$$\phi(x, 0) = 50 \quad \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2},$$
$$\phi\left(\frac{-\pi}{2}, y\right) = 100 \quad \text{for } y > 0.$$

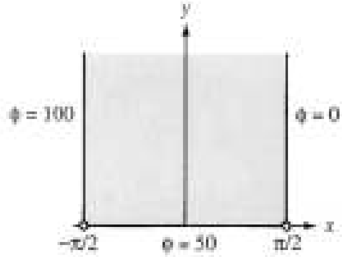


FIGURE 10.42 Accompanies Exercise 4.

5. Find the electrostatic potential $\phi(x, y)$ in the domain D in the half plane $\mathrm{Re}(z) > 0$ that lies to the left of the hyperbola $2x^2 - 2y^2 = 1$ and satisfies the boundary values as shown in Figure 10.43. *Hint*: Use $w = \mathrm{Arcsin}\, z$.

$$\phi(0, y) = 50 \quad \text{for all } y,$$
$$\phi(x, y) = 100 \quad \text{when } 2x^2 - 2y^2 = 1.$$

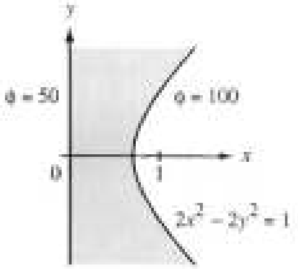


FIGURE 10.43 Accompanies Exercise 5.

6. Find the electrostatic potential $\phi(x, y)$ in the infinite strip $0 < x < \pi/2$ that satisfies the boundary values as shown in Figure 10.44. *Hint*: Use $w = \sin z$.

$$\begin{aligned} \phi(0, y) &= 100 && \text{for } y > 0, \\ \phi\left(\frac{\pi}{2}, y\right) &= 0 && \text{for all } y, \\ \phi(0, y) &= -100 && \text{for } y < 0. \end{aligned}$$

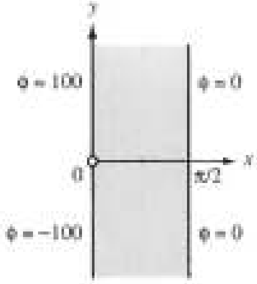
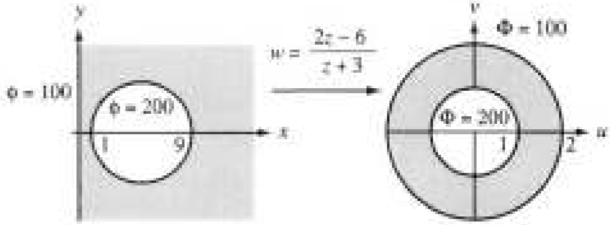


FIGURE 10.44 Accompanies Exercise 6.

7. **(a)** Show that the conformal mapping $w = S(z) = (2z - 6)/(z + 3)$ maps the domain D that is the portion of the right half plane $\text{Re}(z) > 0$ that lies exterior to the circle $|z - 5| = 4$ onto the annulus $1 < |w| < 2$.
   (b) Find the electrostatic potential $\phi(x, y)$ in the domain D that satisfies the boundary values as shown in Figure 10.45.

$$\phi(0, y) = 100 \quad \text{for all } y, \quad \phi(x, y) = 200 \quad \text{when } |z - 5| = 4.$$



8. **(a)** Show that the conformal mapping $w = S(z) = (z - 10)/(2z - 5)$ maps the domain D that is the portion of the disk $|z| < 5$ that lies outside the circle $|z - 2| = 2$ onto the annulus $1 < |w| < 2$.

(b) Find the electrostatic potential $\phi(x, y)$ in the domain D that satisfies the boundary values as shown in Figure 10.46.

$$\phi(x, y) = 100 \quad \text{when } |z| = 5, \quad \phi(x, y) = 200 \quad \text{when } |z - 2| = 2.$$

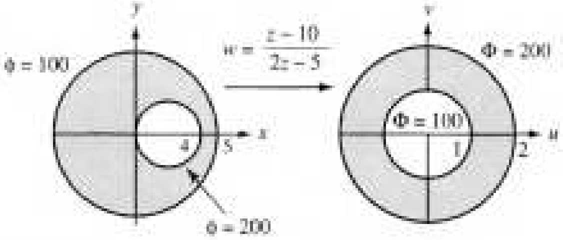


FIGURE 10.46 Accompanies Exercise 8.

10.7 Two-Dimensional Fluid Flow

Suppose that a fluid flows over the complex plane and that the velocity at the point $z = x + iy$ is given by the velocity vector

$$(1) \quad \mathbf{V}(x, y) = p(x, y) + iq(x, y).$$

We also require that the velocity does not depend on time and that the components $p(x, y)$ and $q(x, y)$ have continuous partial derivatives. The divergence of the vector field in equation (1) is given by

$$(2) \quad \text{div } \mathbf{V}(x, y) = p_x(x, y) + q_y(x, y)$$

and is a measure of the extent to which the velocity field diverges near the point. We will consider only fluid flows for which the divergence is zero. This is more precisely characterized by requiring that the net flow through any simply closed contour be identically zero.

If we consider the flow out of the small rectangle in Figure 10.47, then the rate of outward flow equals the line integral of the exterior normal component of $\mathbf{V}(x, y)$ taken over the sides of the rectangle. The exterior normal component is given by $-q$ on the bottom edge, p on the right edge, q on the top edge, and $-p$ on the left edge. Integrating and setting the resulting net flow equal to zero yields

$$(3) \quad \int_y^{y+\Delta y} [p(x + \Delta x, t) - p(x, t)]\, dt$$
$$+ \int_x^{x+\Delta x} [q(t, y + \Delta y) - q(t, y)]\, dt = 0.$$

is an analytic function, then the family of curves

(21) $\{\psi(x, y) = K_2\}$

represents the streamlines of a fluid flow.

The boundary condition for an ideal fluid flow is that **V** should be parallel to the boundary curve containing the fluid (the fluid flows parallel to the walls of a containing vessel). This means that if equation (20) is the complex potential for the flow, then the boundary curve must be given by $\psi(x, y) = K$ for some constant K; that is, the boundary curve must be a streamline.

Theorem 10.5 (Invariance of Flow) *Let*

(22) $F_1(w) = \Phi(u, v) + i\Psi(u, v)$

denote the complex potential for a fluid flow in a domain G in the w plane where the velocity is

(23) $\mathbf{V}_1(u, v) = \overline{F_1'(w)}$.

If the function

(24) $w = S(z) = u(x, y) + iv(x, y)$

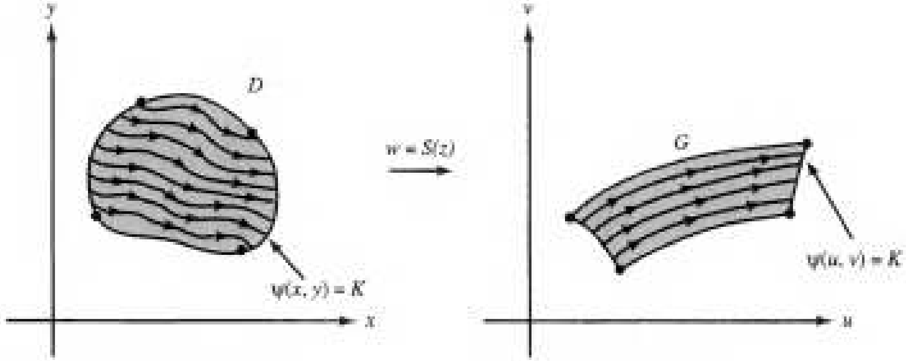
is a one-to-one conformal mapping from a domain D in the z plane onto G, then the composite function

(25) $F_2(z) = F_1(S(z)) = \Phi(u(x, y), v(x, y)) + i\Psi(u(x, y), v(x, y))$

is the complex potential for a fluid flow in D where the velocity is

(26) $\mathbf{V}_2(x, y) = \overline{F_2'(z)}$.

The situation is shown in Figure 10.48.



(a) Fluid flow in the z plane.

(b) Fluid flow in the w plane.

FIGURE 10.48 The image of a fluid flow under conformal mapping.

EXAMPLE 10.23 Consider the complex potential $F(z) = (A/2)z^2$, where A is a positive real number. The velocity potential and stream function are given by

$$(33) \quad \phi(x, y) = \frac{A}{2}(x^2 - y^2) \quad \text{and} \quad \psi(x, y) = Axy,$$

respectively. The streamlines $\psi(x, y) =$ constant form a family of hyperbolas with asymptotes along the coordinate axes. The velocity vector $\mathbf{V} = A\bar{z}$ indicates that in the upper half plane $\text{Im}(z) > 0$ the fluid flows down along the streamlines and spreads out along the x axis. This depicts the flow against a wall and is illustrated in Figure 10.50.

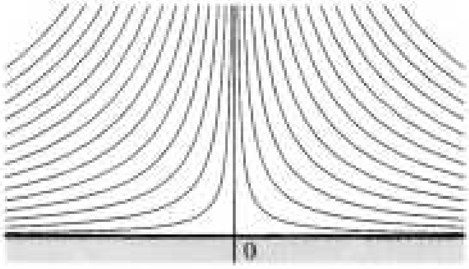


FIGURE 10.50 The fluid flow with complex potential $F(z) = (A/2)z^2$.

EXAMPLE 10.24 Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the unit circle $|z| = 1$.

Solution We will use the fact that the conformal mapping

$$(34) \quad w = S(z) = z + \frac{1}{z}$$

maps the domain $D = \{z: |z| < 1\}$ one-to-one and onto the w plane slit along the segment $-2 \le u \le 2$, $v = 0$. The complex potential for a uniform horizontal flow parallel to this slit in the w plane is

$$(35) \quad F_1(w) = Aw,$$

where A is a positive real number. The stream function for the flow in the w plane is $\psi(u, v) = Av$ so that the slit lies along the streamline $\Psi(u, v) = 0$.

The composite function $F_2(z) = F_1(S(z))$ will determine a fluid flow in the domain D where the complex potential is

$$(36) \quad F_2(z) = A\left(z + \frac{1}{z}\right), \quad \text{where } A > 0.$$

Polar coordinates can be used to express $F_2(z)$ by the equation

$$(37)\quad F_2(z) = A\left(r + \frac{1}{r}\right)\cos\theta + iA\left(r - \frac{1}{r}\right)\sin\theta.$$

The streamline $\psi(r, \theta) = A(r - 1/r)\sin\theta = 0$ consists of the rays

$$(38)\quad r > 1, \theta = 0 \quad\text{and}\quad r > 1, \theta = \pi$$

along the x axis and the curve $r - 1/r = 0$, which is easily seen to be the unit circle $r = 1$. This shows that the unit circle can be considered as a boundary curve for the fluid flow.

Since the approximation $F_2(z) = A(z + 1/z) \approx Az$ is valid for large values of z, we see that the flow is approximated by a uniform horizontal flow with speed $|\mathbf{V}| = A$ at points that are distant from the origin. The streamlines $\psi(x, y) =$ constant and their images $\Psi(u, v) =$ constant under the mapping $w = S(z) = z + 1/z$ are illustrated in Figure 10.51.

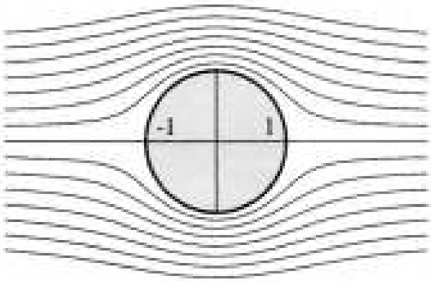


FIGURE 10.51 Fluid flow around a circle.

EXAMPLE 10.25 Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the segment from $-i$ to i.

Solution We will use the conformal mapping

$$(39)\quad w = S(z) = (z^2 + 1)^{1/2} = (z + i)^{1/2}(z - i)^{1/2},$$

where the branch of the square root of $Z = z \pm i$ in each factor is $Z^{1/2} = R^{1/2}e^{i\theta/2}$, where $R = |Z|$, and $\theta = \arg Z$, where $-\pi/2 < \arg Z \le 3\pi/2$. The function given by $w = S(z)$ is a one-to-one conformal mapping of the domain D consisting of the z plane slit along the segment $x = 0, -1 \le y \le 1$ onto the domain G consisting of the w plane slit along the segment $-1 \le u \le 1, v = 0$.

EXERCISES FOR SECTION 10.7

1. Consider the ideal fluid flow where the complex potential is $F(z) = A(z + 1/z)$, where A is a positive real number.
   - **(a)** Show that the velocity vector at the point $(1, \theta)$, $z = re^{i\theta}$ on the unit circle is given by $\mathbf{V}(1, \theta) = A(1 - \cos 2\theta - i \sin 2\theta)$.
   - **(b)** Show that the velocity vector $\mathbf{V}(1, \theta)$ is tangent to the unit circle $|z| = 1$ at all points except -1 and $+1$. *Hint:* Show that $\mathbf{V} \cdot \mathbf{P} = 0$, where $\mathbf{P} = \cos\theta + i \sin\theta$.
   - **(c)** Show that the speed at the point $(1, \theta)$ on the unit circle is given by $|\mathbf{V}| = 2A|\sin\theta|$ and that the speed attains the maximum of $2A$ at the points $\pm i$ and is zero at the points ± 1. Where is the pressure the greatest?
2. Show that the complex potential $F(z) = ze^{-i\alpha} + e^{i\alpha}/z$ determines the ideal fluid flow around the unit circle $|z| = 1$ where the velocity at points distant from the origin is given approximately by $\mathbf{V} \approx e^{i\alpha}$; that is, the direction of the flow for large values of z is inclined at an angle α with the x axis, as shown in Figure 10.53.

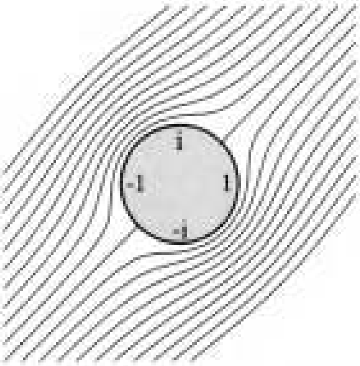


FIGURE 10.53 Accompanies Exercise 2.

3. Consider the ideal fluid flow in the channel bounded by the hyperbolas $xy = 1$ and $xy = 4$ in the first quadrant, where the complex potential is given by $F(z) = (A/2)z^2$ and A is a positive real number.
   - **(a)** Find the speed at each point, and find the point on the boundary where the speed attains a minimum value.
   - **(b)** Where is the pressure greatest?
4. Show that the stream function is given by $\psi(r, \theta) = Ar^3 \sin 3\theta$ for an ideal fluid flow around the angular region $0 < \theta < \pi/3$ indicated in Figure 10.54. Sketch several streamlines of the flow. *Hint:* Use the conformal mapping $w = z^3$.

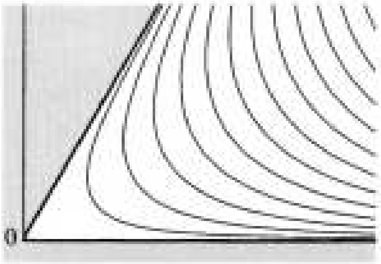


FIGURE 10.54 Accompanies Exercise 4.

5. Consider the ideal fluid flow, where the complex potential is

$$F(z) = Az^{3/2} = Ar^{3/2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right), \text{ where } 0 \le \theta \le 2\pi.$$

   (a) Find the stream function $\psi(r, \theta)$.
   (b) Sketch several streamlines of the flow in the angular region $0 < \theta < 4\pi/3$ as indicated in Figure 10.55.

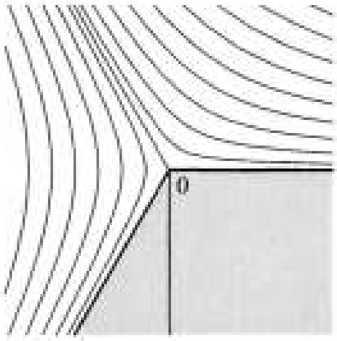


FIGURE 10.55 Accompanies Exercise 5.

6. **(a)** Let $A > 0$. Show that the potential $F(z) = A(z^2 + 1/z^2)$ determines an ideal fluid flow around the domain $r > 1$, $0 < \theta < \pi/2$ indicated in Figure 10.56, which shows the flow around a circle in the first quadrant. *Hint:* Use the conformal mapping $w = z^2$.
   (b) Show that the speed at the point $(1, \theta)$, $z = re^{i\theta}$ on the quarter circle $r = 1$, $0 < \theta < \pi/2$ is given by $\mathbf{V} = 4A|\sin 2\theta|$.
   (c) Determine the stream function for the flow and sketch several streamlines.

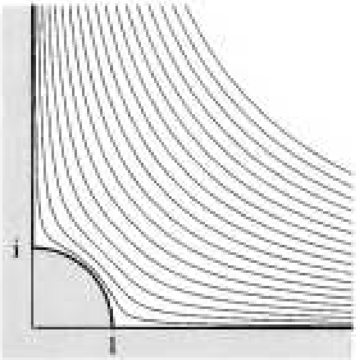


FIGURE 10.56 Accompanies Exercise 6.

7. Show that $F(z) = \sin z$ is the complex potential for the ideal fluid flow inside the semi-infinite strip $-\pi/2 < x < \pi/2$, $y > 0$, as indicated in Figure 10.57. Find the stream function.

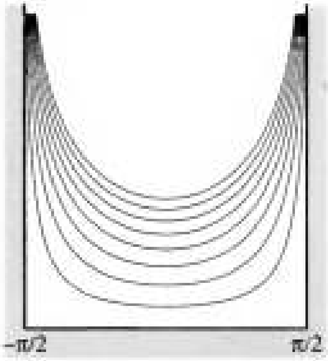


FIGURE 10.57 Accompanies Exercise 7.

8. Let $w = S(z) = \frac{1}{2}[z + (z^2 - 4)^{1/2}]$ denote the branch of the inverse of $z = w + 1/w$ that is a one-to-one mapping of the z plane slit along the segment $-2 \le x \le 2$, $y = 0$ onto the domain $|w| > 1$. Use the complex potential $F_2(w) = we^{-i\alpha} + (e^{i\alpha}/w)$ in the w plane to show that the complex potential $F_1(z) = z\cos\alpha - i(z^2 - 4)^{1/2}\sin\alpha$ determines the ideal fluid flow around the segment $-2 \le x \le 2$, $y = 0$, where the velocity at points distant from the origin is given approximately by $\mathbf{V} = e^{i\alpha}$, as shown in Figure 10.58.
9. (a) Show that the complex potential $F(z) = -i \operatorname{Arcsin} z$ determines the ideal fluid flow through the aperture from -1 to $+1$, as indicated in Figure 10.59.
   (b) Show that the streamline $\psi(x, y) = c$ for the flow is a portion of the hyperbola $(x^2/\sin^2 c) - (y^2/\cos^2 c) = 1$.

region onto the w plane slit along the portion of the real axis $-2 \le u \le 2$, and the points $W_1 = -1$, $W_2 = 0$, $W_3 = -1$, and $W_4 = -\infty$ are mapped onto $w_1 = 0$, $w_2 = 2$, $w_3 = 0$, and $w_4 = -2$, respectively. These three compositions are shown in Figure 10.61.

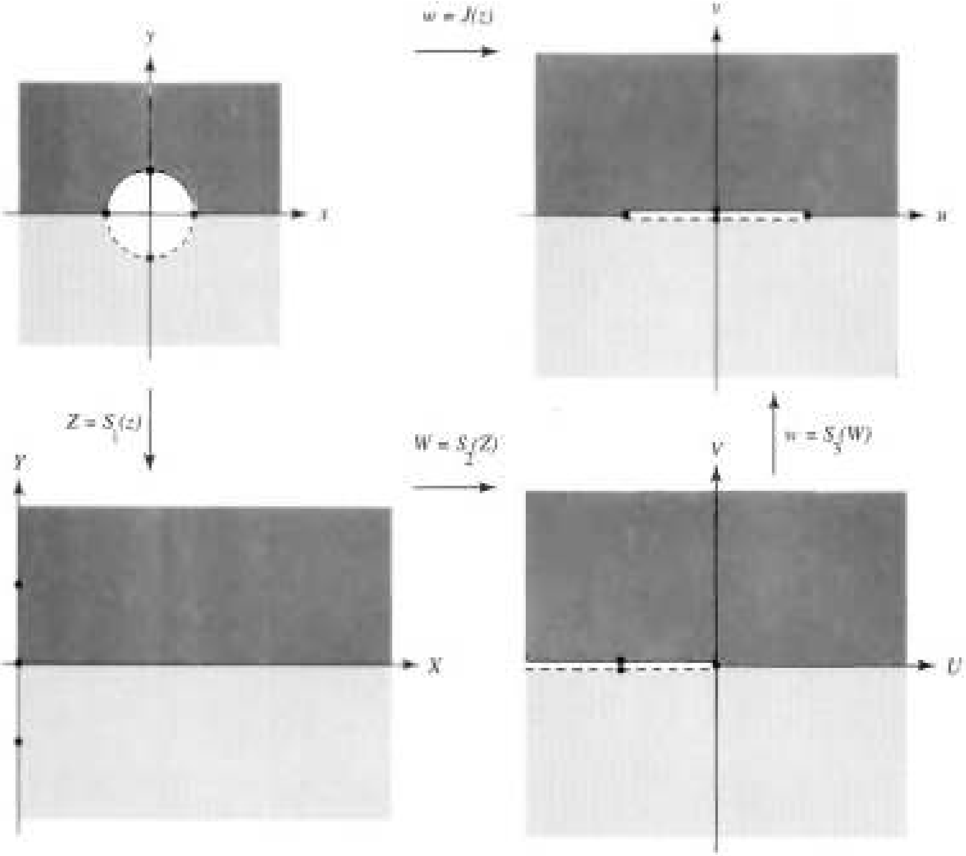


FIGURE 10.61 The composition mappings for $J(z) = S_3(S_2(S_1(z)))$.

The circle C_0 with center $c_0 = ia$ on the imaginary axis passes through the points $z_2 = 1$ and $z_4 = -1$ and has radius $r_0 = \sqrt{1 + a^2}$. If we restrict $0 < a < 1$, then this circle intersects the x axis at the point z_2 with angle $\alpha_0 = \frac{\pi}{2} - \arctan a$, with $\frac{\pi}{4} < \alpha_0 < \frac{\pi}{2}$. We want to track the image of C_0 in the Z, W, and w planes. First, the image of this circle C_0 under $Z = S_1(z)$ is the line L_0 that passes through the origin and is inclined at the angle α_0. Second, the function $W = S_2(Z)$ maps the

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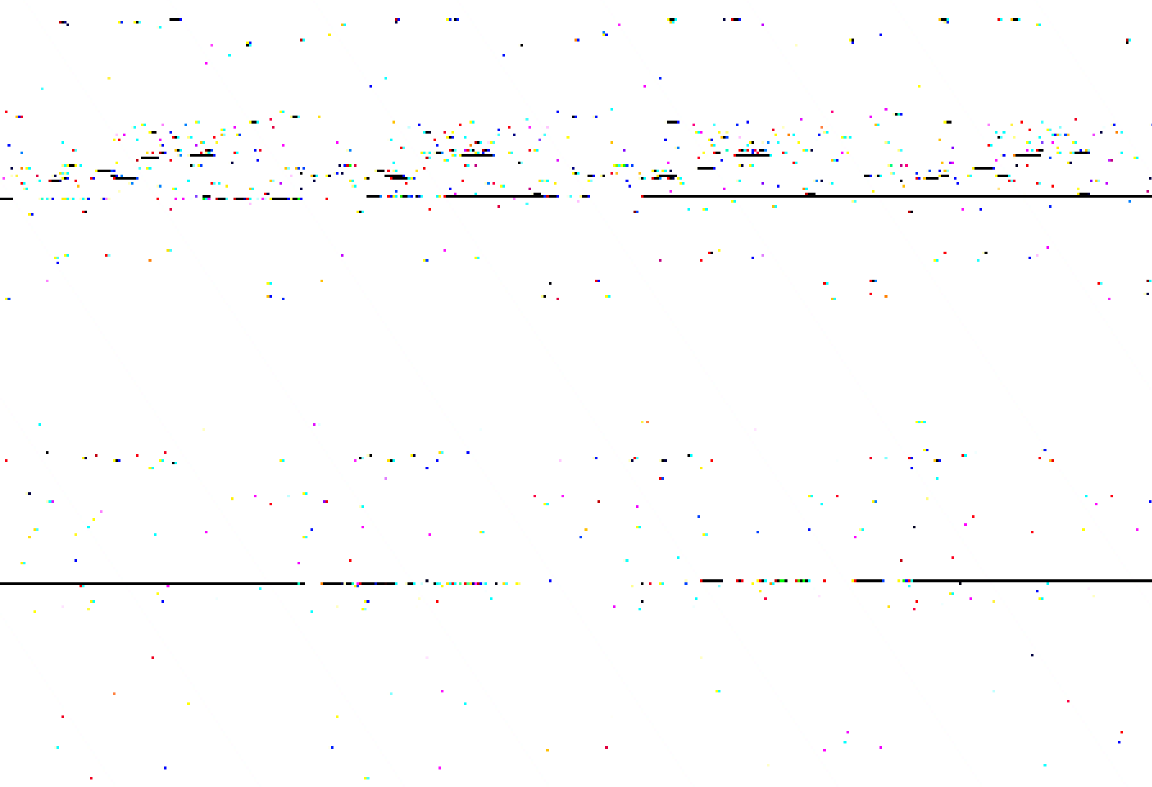

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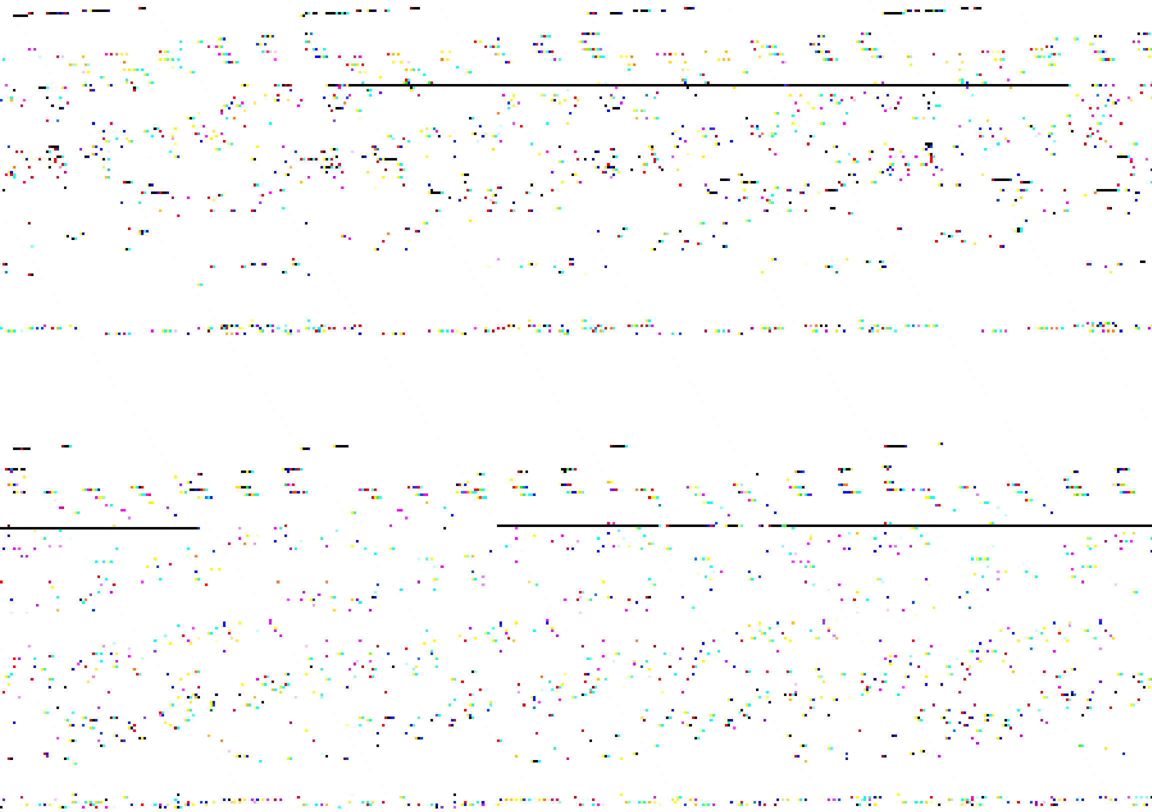


$w = f(z)$ so that the line segment $x_{j-1} < x < x_j$ is mapped onto the edge of P that lies between the points $w_{j-1} = f(x_{j-1})$ and $w_j = f(x_j)$. Since the amount of rotation is determined by $\arg f'(x)$, formula (5) specifies $f'(z)$ in terms of the values x_j and the amount of rotation α_j that is required at the vertex $f(x_j)$.

If we let $x_0 = -\infty$ and $x_n = \infty$, then, for values of x that lie in the interval $x_{j-1} < x < x_j$, the amount of rotation is

$$(8) \quad \arg f'(x) = \arg A - \frac{1}{\pi}[\alpha_1 \arg(x - x_1) + \alpha_2 \arg(x - x_2) + \cdots + \alpha_{n-1} \arg(x - x_{n-1})].$$

Since $\operatorname{Arg}(x - x_k) = 0$ for $1 \le k < j$ and $\operatorname{Arg}(x - x_k) = \pi$ for $j \le k \le n - 1$, we can write equation (8) as

$$(9) \quad \arg f'(x) = \arg A - \alpha_j - \alpha_{j+1} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the tangent vector $\mathbf{T}_j$ to the polygon P at the point $w = f(x)$ for $x_{j-1} < x < x_j$ is

$$(10) \quad \gamma_j = \operatorname{Arg} A - \alpha_j - \alpha_{j+1} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the tangent vector $\mathbf{T}_{j+1}$ to the polygon P at the point $w = f(x)$ for $x_j < x < x_{j+1}$ is

$$(11) \quad \gamma_{j+1} = \operatorname{Arg} A - \alpha_{j+1} - \alpha_{j+2} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the vector tangent to the polygon P jumps abruptly by the amount α_j as the point $w = f(x)$ moves along the side $\overparen{w_{j-1}w_j}$ through the vertex w_j to the side $\overparen{w_jw_{j+1}}$. Therefore the exterior angle to the polygon P at the vertex w_j is given by the angle α_j and satisfies the inequality $-\pi < \alpha_j < \pi$ for $j = 1, 2, \ldots, n - 1$. Since the sum of the exterior angles of a polygon equals 2π, we have $\alpha_n = 2\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}$ so that only $n - 1$ angles need to be specified. This case with $n = 5$ is indicated in Figure 10.70.

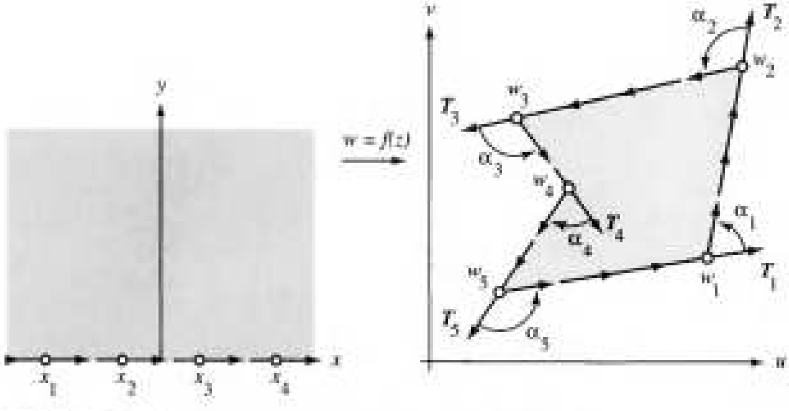


FIGURE 10.70 A Schwarz-Christoffel mapping with $n = 5$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_4 > \pi$.

If the case $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \le \pi$ occurs, then $\alpha_n > \pi$, and the vertices $w_1, w_2, \ldots, w_n$ cannot form a closed polygon. For this case, formulas (5) and (6) will determine a mapping from the upper half plane $\operatorname{Im}(z) > 0$ onto an infinite region in the w plane where the vertex w_n is at infinity. The case $n = 5$ is illustrated in Figure 10.71.

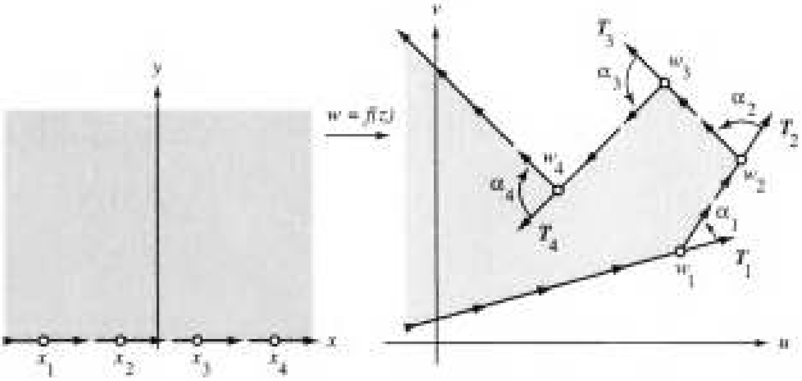


FIGURE 10.71 A Schwarz-Christoffel mapping with $n = 5$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_4 \le \pi$.

Formula (6) gives a representation for f in terms of an indefinite integral. It is important to note that these integrals do not represent elementary functions unless the image is an infinite region. Also, the integral will involve a multivalued function, and a specific branch must be selected to fit the boundary values specified in the problem. Table 10.2 is useful for our purposes.

TABLE 10.2 Indefinite Integrals

$$\int \frac{dz}{(z^2-1)^{1/2}} = i \arcsin z = \log(z + (z^2-1)^{1/2}) - \frac{i\pi}{2}$$

$$\int \frac{dz}{z^2+1} = \arctan z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$$

$$\int \frac{dz}{z(z^2-1)^{1/2}} = -\arcsin \frac{1}{z} = i \log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right]$$

$$\int \frac{dz}{z(z+1)^{1/2}} = -2 \operatorname{arctanh}[(z+1)^{1/2}] = \log\left[\frac{1-(z+1)^{1/2}}{1+(z+1)^{1/2}}\right]$$

$$\int (1-z^2)^{1/2}\, dz = \frac{1}{2}[z(1-z^2)^{1/2} + \arcsin z]$$

$$= \frac{i}{2}[z(z^2-1)^{1/2} + \log(z + (z^2-1)^{1/2})]$$

EXAMPLE 10.26 Use the Schwarz-Christoffel formula to verify that the function $w = f(z) = \text{Arcsin } z$ maps the upper half plane $\text{Im}(z) > 0$ onto the semi-infinite strip $-\pi/2 < u < \pi/2$, $v > 0$ shown in Figure 10.72.

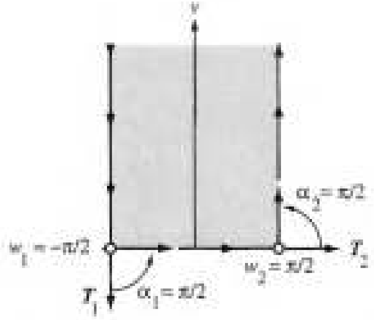


FIGURE 10.72 The region in Example 10.26.

Solution If we choose $x_1 = -1$, $x_2 = 1$, $w_1 = -\pi/2$, and $w_2 = \pi/2$, then $\alpha_1 = \pi/2$ and $\alpha_2 = \pi/2$, and equation (5) for $f'(z)$ becomes

$$(12)\quad f'(z) = A(z+1)^{-(\pi/2)/\pi}(z-1)^{-(\pi/2)/\pi} = \frac{A}{(z^2-1)^{1/2}}.$$

Using Table 10.2 we see that the solution to equation (12) is

$$(13)\quad f(z) = Ai \text{ Arcsin } z + B.$$

Using the image values $f(-1) = -\pi/2$ and $f(1) = \pi/2$, we obtain the system

$$(14)\quad \frac{-\pi}{2} = A\frac{-i\pi}{2} + B \quad \text{and} \quad \frac{\pi}{2} = A\frac{i\pi}{2} + B,$$

which can be solved to obtain $B = 0$ and $A = -i$. Hence the required function is

$$(15)\quad f(z) = \text{Arcsin } z.$$

EXAMPLE 10.27 Verify that $w = f(z) = (z^2 - 1)^{1/2}$ maps the upper half plane $\text{Im}(z) > 0$ onto the upper half plane $\text{Im}(w) > 0$ slit along the segment from 0 to i.

Solution If we choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = -d$, $w_2 = i$, and $w_3 = d$, then we see that the formula

$$(16)\quad g'(z) = A(z+1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z-1)^{-\alpha_3/\pi}$$

will determine a mapping $w = g(z)$ from the upper half plane $\text{Im}(z) > 0$ onto the portion of the upper half plane $\text{Im}(w) > 0$ that lies outside the triangle with vertices $\pm d$, i as indicated in Figure 10.73(a). If we let $d \to 0$, then $w_1 \to 0$, $w_3 \to 0$,

$\alpha_1 \to \pi/2$, $\alpha_2 \to -\pi$, and $\alpha_3 \to \pi/2$. The limiting formula for the derivative in equation (16) becomes

$$(17)\quad f'(z) = A(z+1)^{-1/2}(z)(z-1)^{-1/2},$$

which will determine a mapping $w = f(z)$ from the upper half plane $\operatorname{Im}(z) > 0$ onto the upper half plane $\operatorname{Im}(w) > 0$ slit from 0 to i as indicated in Figure 10.73(b). An easy computation reveals that $f(z)$ is given by

$$(18)\quad f(z) = A\int \frac{z\,dz}{(z^2-1)^{1/2}} = A(z^2-1)^{1/2} + B,$$

and the boundary values $f(\pm 1) = 0$ and $f(0) = i$ lead to the solution

$$(19)\quad f(z) = (z^2-1)^{1/2}.$$

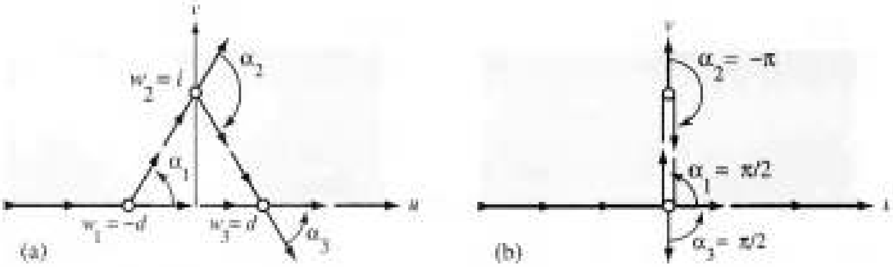


FIGURE 10.73 The regions in Example 10.27.

EXAMPLE 10.28

Show that the function

$$(20)\quad w = f(z) = \frac{1}{\pi}\operatorname{Arcsin} z + \frac{i}{\pi}\operatorname{Arcsin}\frac{1}{z} + \frac{1+i}{2}$$

maps the upper half plane $\operatorname{Im}(z) > 0$ onto the right angle channel in the first quadrant, which is bounded by the coordinate axes and the rays $x \geq 1$, $y = 1$ and $y \geq 1$, $x = 1$ in Figure 10.74(b).

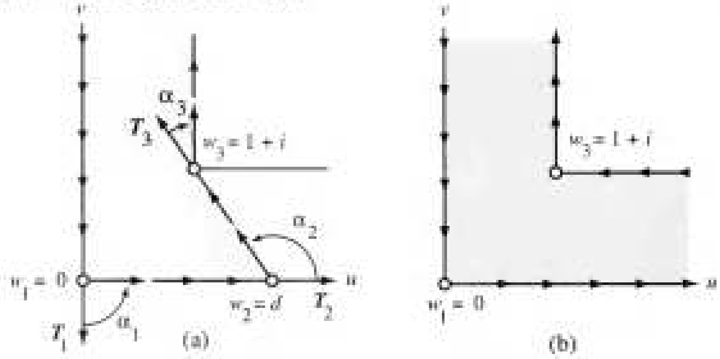


FIGURE 10.74 The regions in Example 10.28.

Solution If we choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = d$, and $w_3 = 1 + i$, then the formula

$$g'(z) = A_1(z+1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z-1)^{-\alpha_3/\pi} \tag{21}$$

will determine a mapping of the upper half plane onto the domain indicated in Figure 10.74(a).

If we let $d \to \infty$, then $\alpha_2 \to \pi$ and $\alpha_3 \to -\pi/2$. Then the limiting formula for the derivative in equation (21) becomes

$$\begin{aligned} f'(z) &= A_1(z+1)^{-(\pi/2)/\pi}(z)^{-(\pi)/\pi}(z-1)^{-(-\pi/2)/\pi} \\ &= A_1 \frac{1}{z}\frac{(z-1)^{1/2}}{(z+1)^{1/2}} = A_1 \frac{z-1}{z(z^2-1)^{1/2}} = A \frac{z-1}{z(1-z^2)^{1/2}}, \end{aligned} \tag{22}$$

where $A = -iA_1$, which will determine a mapping $w = f(z)$ from the upper half plane onto the channel as indicated in Figure 10.74(b). Using Table 10.2, we obtain

$$\begin{aligned} f(z) &= A\left[\int \frac{dz}{(1-z^2)^{1/2}} - i\int \frac{dz}{z(z^2-1)^{1/2}}\right] \\ &= A\left[\arcsin z + i \arcsin \frac{1}{z}\right] + B. \end{aligned} \tag{23}$$

If the principal branch of the inverse sine function is used, then the boundary values $f(-1) = 0$ and $f(1) = 1 + i$ lead to the system

$$A\left[\frac{-\pi}{2} + i\left(\frac{-\pi}{2}\right)\right] + B = 0, \quad A\left[\frac{\pi}{2} + i\left(\frac{\pi}{2}\right)\right] + B = 1 + i,$$

which can be solved to obtain $A = 1/\pi$ and $B = (1 + i)/2$. Hence the required solution is

$$w = f(z) = \frac{1}{\pi}\operatorname{Arcsin} z + \frac{i}{\pi}\operatorname{Arcsin}\frac{1}{z} + \frac{1+i}{2}. \tag{24}$$

EXERCISES FOR SECTION 10.9

1. Let a and K be real constants with $0 < K < 2$. Use the Schwarz-Christoffel formula to show that the function $w = f(z) = (z - a)^K$ maps the upper half plane $\operatorname{Im}(z) > 0$ onto the sector $0 < \arg w < K\pi$, shown in Figure 10.75.

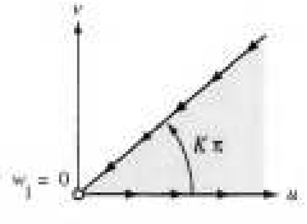


FIGURE 10.75 Accompanies Exercise 1.

2. Let a be a real constant. Use the Schwarz-Christoffel formula to show that the function $w = f(z) = \text{Log}(z - a)$ maps the upper half plane $\text{Im}(z) > 0$ onto the infinite strip $0 < v < \pi$ in Figure 10.76. *Hint:* Set $x_1 = a - 1$, $x_2 = a$, $w_1 = i\pi$, $w_2 = -d$, and let $d \to \infty$.

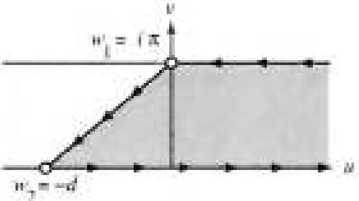


FIGURE 10.76 Accompanies Exercise 2.

3. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = \frac{1}{\pi}\left((z^2 - 1)^{1/2} + \text{Log}[z + (z^2 - 1)^{1/2}]\right) - i$$

maps the upper half plane onto the domain indicated in Figure 10.77. *Hint:* Set $x_1 = -1$, $x_2 = 1$, $w_1 = 0$, and $w_2 = -i$.

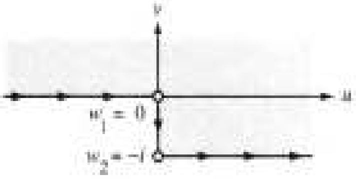


FIGURE 10.77 Accompanies Exercise 3.

4. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = \frac{2}{\pi}\left[(z^2 - 1)^{1/2} + \text{Arcsin}\,\frac{1}{z}\right]$$

maps the upper half plane onto the domain indicated in Figure 10.78. *Hint:* Set $x_1 = w_1 = -1$, $x_2 = 0$, $x_3 = w_3 = 1$, and $w_2 = -id$ and let $d \to \infty$.

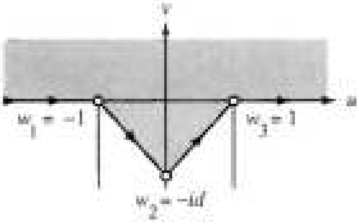


FIGURE 10.78 Accompanies Exercise 4.

5. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = \frac{1}{2}\log(z^2 - 1) = \mathrm{Log}[(z^2 - 1)^{1/2}]$$

maps the upper half plane $\mathrm{Im}(z) > 0$ onto the infinite strip $0 < v < \pi$ slit along the ray $u \le 0$, $v = \pi/2$, see Figure 10.79. *Hint:* Set $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = i\pi - d$, $w_2 = i\pi/2$, and $w_3 = -d$ and let $d \to \infty$.

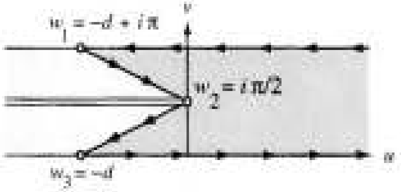


FIGURE 10.79 Accompanies Exercise 5.

6. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = \frac{-2}{\pi}[z(1 - z^2)^{1/2} + \mathrm{Arcsin}\, z]$$

maps the upper half plane onto the domain indicated in Figure 10.80. *Hint:* Set $x_1 = -1$, $x_2 = 1$, $w_1 = 1$, and $w_2 = -1$.

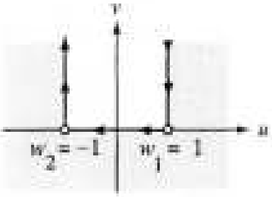


FIGURE 10.80 Accompanies Exercise 6.

7. Use the Schwarz-Christoffel formula to show that the function $w = f(z) = z + \mathrm{Log}\, z$ maps the upper half plane $\mathrm{Im}(z) > 0$ onto the upper half plane $\mathrm{Im}(w) > 0$ slit along the ray $u \le -1$, $v = \pi$, shown in Figure 10.81. *Hint:* Set $x_1 = -1$, $x_2 = 0$, $w_1 = -1 + i\pi$, and $w_2 = -d$ and let $d \to \infty$.

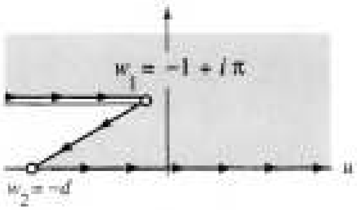


FIGURE 10.81 Accompanies Exercise 7.

8. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = 2(z+1)^{1/2} + \operatorname{Log}\left[\frac{1-(z+1)^{1/2}}{1+(z+1)^{1/2}}\right] + i\pi$$

maps the upper half plane onto the domain indicated in Figure 10.82. *Hint:* Set $x_1 = -1$, $x_2 = 0$, $w_1 = i\pi$, and $w_2 = -d$ and let $d \to \infty$.

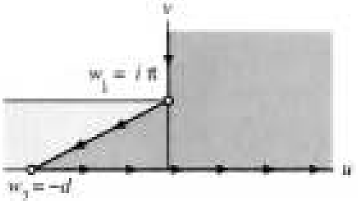


FIGURE 10.82 Accompanies Exercise 8.

9. Show that the function $w = f(z) = (z-1)^{\alpha}[1 + \alpha z/(1-\alpha)]^{1-\alpha}$ maps the upper half plane $\operatorname{Im}(z) > 0$ onto the upper half plane $\operatorname{Im}(w) > 0$ slit along the segment from 0 to $e^{i\alpha\pi}$, as shown in Figure 10.83. *Hint:* Show that $f'(z) = A[z + (1-\alpha)/\alpha]^{-\alpha}(z)(z-1)^{\alpha-1}$.

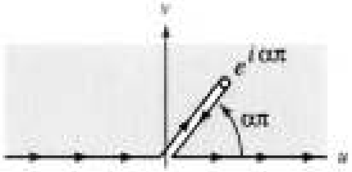


FIGURE 10.83 Accompanies Exercise 9.

10. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = 4(z+1)^{1/4} + \log\left[\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1}\right] + i\log\left[\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}\right]$$

maps the upper half plane onto the domain indicated in Figure 10.84. *Hint:* Set $z_1 = -1$, $z_2 = 0$, $w_1 = i\pi$, and $w_2 = -d$ and let $d \to \infty$. Use the change of variable $z + 1 = s^4$ in the resulting integral.

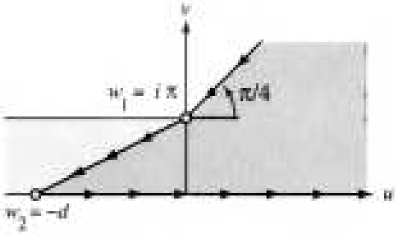


FIGURE 10.84 Accompanies Exercise 10.

11. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = \frac{-i}{2} z^{1/2}(z - 3)$$

maps the upper half plane onto the domain indicated in Figure 10.85. *Hint:* Set $x_1 = 0$, $x_2 = 1$, $w_1 = -d$, and $w_2 = i$ and let $d \to 0$.

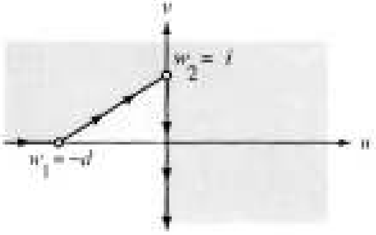


FIGURE 10.85 Accompanies Exercise 11.

12. Show that the function

$$w = f(z) = \int \frac{dz}{(1 - z^2)^{3/4}}$$

maps the upper half plane $\text{Im}(z) > 0$ onto a right triangle with angles $\pi/2$, $\pi/4$, and $\pi/4$.

13. Show that the function

$$w = f(z) = \int \frac{dz}{(1 - z^2)^{2/3}}$$

maps the upper half plane onto an equilateral triangle.

14. Show that the function

$$w = f(z) = \int \frac{dz}{(z - z^3)^{1/2}}$$

maps the upper half plane onto a square.

15. Use the Schwarz-Christoffel formula to show that the function

$$w = f(z) = 2(z + 1)^{1/2} - \text{Log}\left[\frac{1 - (z + 1)^{1/2}}{1 + (z + 1)^{1/2}}\right]$$

maps the upper half plane $\text{Im}(z) > 0$ onto the domain indicated in Figure 10.86. *Hint:* Set $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = d$, and $w_3 = 2\sqrt{2} - 2\ln(\sqrt{2} - 1) + i\pi$ and let $d \to \infty$.

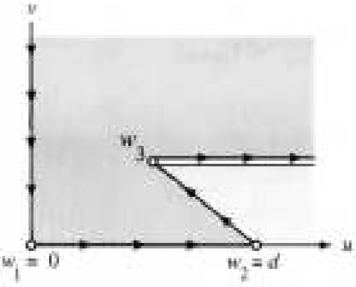


FIGURE 10.86 Accompanies Exercise 15.

16. Write a report on the Schwarz-Christoffel transformation. Include ideas and examples not mentioned in the text. Resources include bibliographical items 93, 159, and 164.

10.10 Image of a Fluid Flow

We have already examined several two-dimensional fluid flows and have discovered that the image of a flow under a conformal transformation is a flow. The conformal mapping $w = f(z) = u(x, y) + iv(x, y)$, which is obtained by using the Schwarz-Christoffel formula, will allow us to find the streamlines for flows in domains in the w plane that are bounded by straight line segments.

The first technique is finding the image of a fluid flowing horizontally from left to right across the upper half plane $\mathrm{Im}(z) > 0$. The image of the streamline $-\infty < t < \infty$, $y = c$ will be a streamline given by the parametric equations

$$(1)\quad u = u(t, c), \quad v = v(t, c) \quad \text{for } -\infty < t < \infty$$

and will be oriented in the counterclockwise (positive) sense. The streamline $u = u(t, 0)$, $v = v(t, 0)$ is considered to be a boundary wall for a containing vessel for the fluid flow.

EXAMPLE 10.29 Consider the conformal mapping

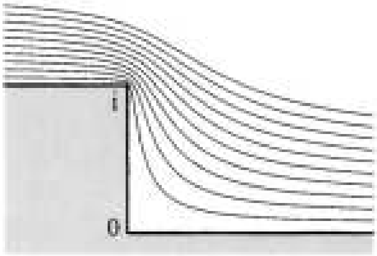
$$(2)\quad w = f(z) = \frac{1}{\pi}[(z^2 - 1)^{1/2} + \mathrm{Log}(z + (z^2 - 1)^{1/2})],$$

which is obtained by using the Schwarz-Christoffel formula, to map the upper half plane $\mathrm{Im}(z) > 0$ onto the domain in the w plane that lies above the boundary curve consisting of the rays $u \le 0$, $v = 1$ and $u \ge 0$, $v = 0$ and the segment $u = 0$, $-1 \le v \le 0$.

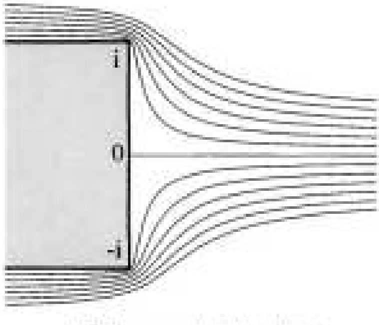
The image of horizontal streamlines in the z plane are curves in the w plane given by the parametric equation

$$(3)\quad w = f(t + ic) = \frac{1}{\pi}(t^2 - c^2 - 1 + i2ct)^{1/2} + \frac{1}{\pi}\operatorname{Log}[t + ic + (t^2 - c^2 - 1 + i2ct)^{1/2}]$$

for $-\infty < t < \infty$. The new flow is that of a step in the bed of a deep stream and is illustrated in Figure 10.87(a). The function $w = f(z)$ is also defined for values of z in the lower half plane, and the images of horizontal streamlines that lie above or below the x axis are mapped onto streamlines that flow past a long rectangular obstacle. This is illustrated in Figure 10.87(b).



(a) Flow over a step



(b) Flow around a blunt object.

FIGURE 10.87 Accompanies Example 10.29.

EXERCISES FOR SECTION 10.10

For Exercises 1–4, use the Schwarz-Christoffel formula to find a conformal mapping $w = f(z)$ that will map the flow in the upper half plane $\text{Im}(z) > 0$ onto the flow indicated in each of the following figures.

1.

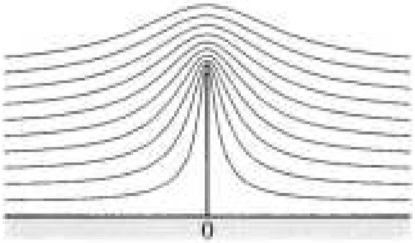


FIGURE 10.88 Accompanies Exercise 1.

2.

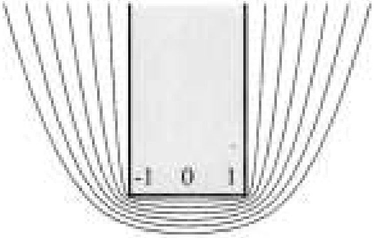
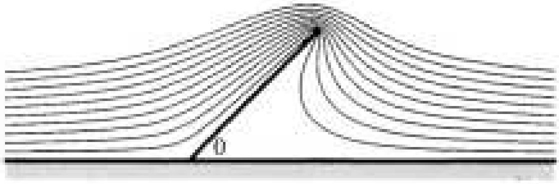
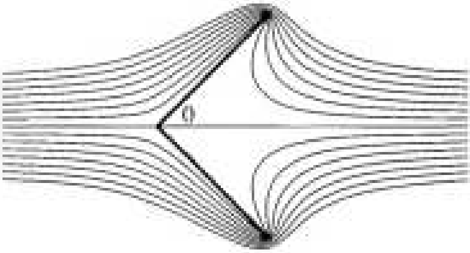


FIGURE 10.89 Accompanies Exercise 2.

3.



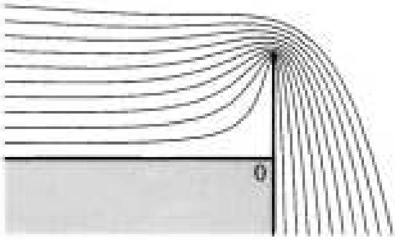
(a) Flow around an inclined segment.



(b) Flow around a V-shape.

FIGURE 10.90 Accompanies Exercise 3.

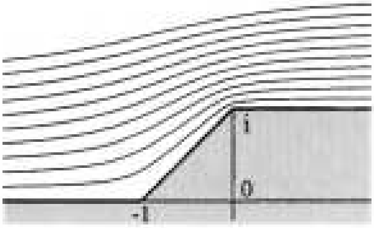
4.



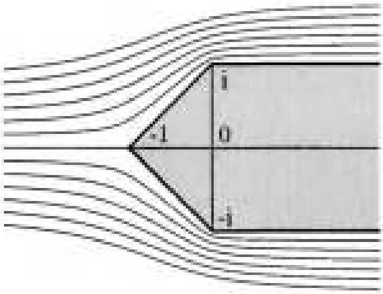
Flow over a dam.

FIGURE 10.91 Accompanies Exercise 4.

5. Use the Schwarz-Christoffel formula, and find an expression for $f'(z)$ for the transformation $w = f(z)$ that will map the upper half plane $\text{Im}(z) > 0$ onto the flow indicated in Figure 10.92(a). Extend the flow to the one indicated in Figure 10.92(b).



(a) Flow up an inclined step.



(b) Flow around a pointed object.

FIGURE 10.92 Accompanies Exercise 5.

10.11 Sources and Sinks

If the two-dimensional motion of an ideal fluid consists of an outward radial flow from a point and is symmetrical in all directions, then the point is called a *simple source*. A source at the origin can be considered as a line perpendicular to the z plane along which fluid is being created. If the rate of emission of volume of fluid per unit length is $2\pi m$, then the origin is said to be a source of strength m, the complex potential for the flow is

(1) $F(z) = m \log z,$

and the velocity **V** at the point (x, y) is given by

(2) $\mathbf{V}(x, y) = \overline{F'(z)} = \dfrac{m}{\bar{z}}.$

For fluid flows a sink is a negative source and is a point of inward radial flow at which the fluid is considered to be absorbed or annihilated. Sources and sinks for flows are illustrated in Figure 10.93.

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$u \le -1$, $v = 0$ and $u \ge 1$, $v = 0$, and the image of the source at $z_0 = 0$ is a source located at $w_0 = 0$. It is easy to see that the complex potential

(21) $F_1(w) = \log w$

will determine a fluid flow in the w plane past the boundary curves $u \le -1$, $v = 0$ and $u \ge 1$, $v = 0$, which lie along streamlines of the flow. Therefore the complex potential for the fluid flow in the infinite strip in the z plane is

(22) $F_2(z) = \log(\sin z)$.

Several streamlines for the flow are illustrated in Figure 10.96.

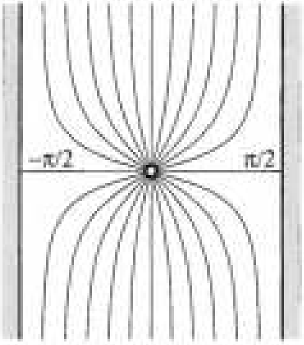

FIGURE 10.96 A source in the center of a strip.

EXAMPLE 10.33 Suppose that the lines $x = \pm\pi/2$ are considered as walls of a containing vessel for the fluid flow produced by a source of unit strength located at the point $z_1 = \pi/2$ and a sink of unit strength located at the point $z_2 = -\pi/2$. The conformal mapping $w = S(z) = \sin z$ maps the infinite strip bounded by the lines $x = \pm\pi/2$ onto the w plane slit along the boundary rays K_1: $u \le -1$, $v = 0$ and K_2: $u \ge 1$, $v = 0$. The image of the source at z_1 is a source at $w_1 = 1$, and the image of the sink at z_2 is a sink at $w_2 = -1$. It is easy to verify that the potential

(23) $F_1(w) = \log\left(\dfrac{w-1}{w+1}\right)$

will determine a fluid flow in the w plane past the boundary curves K_1 and K_2, which lie along streamlines of the flow. Therefore the complex potential for the fluid flow in the infinite strip in the z plane is

(24) $F_2(z) = \log\left(\dfrac{\sin z - 1}{\sin z + 1}\right)$.

Several streamlines for the flow are illustrated in Figure 10.97.

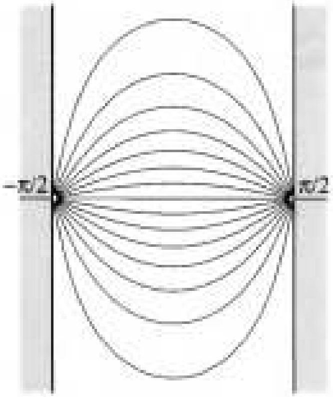

FIGURE 10.97 A source and sink on the edges of a strip.

The technique of transformation of a source can be used to determine the effluence from a channel extending from infinity. In this case, a conformal mapping $w = S(z)$ from the upper half plane $\text{Im}(z) > 0$ is constructed so that the single source located at $z_0 = 0$ is mapped to the point w_0 at infinity that lies along the channel. The streamlines emanating from $z_0 = 0$ in the upper half plane are mapped onto streamlines issuing from the channel.

EXAMPLE 10.34 Consider the conformal mapping

$$(25) \qquad w = S(z) = \frac{2}{\pi}\left[(z^2 - 1)^{1/2} + \text{Arcsin}\,\frac{1}{z}\right],$$

which maps the upper half plane $\text{Im}(z) > 0$ onto the domain consisting of the upper half plane $\text{Im}(w) > 0$ joined to the channel $-1 \le u \le 1$, $v \le 0$. The point $z_0 = 0$ is mapped onto the point $w_0 = -i\infty$ along the channel. Images of the rays $r > 0$, $\theta = \alpha$ are streamlines issuing from the channel as indicated in Figure 10.98.

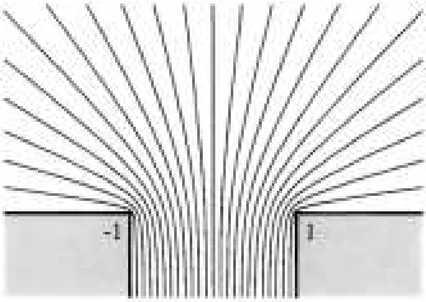

FIGURE 10.98 Effluence from a channel into a half plane.

EXERCISES FOR SECTION 10.11

1. Let the coordinate axes be walls of a containing vessel for a fluid flow in the first quadrant that is produced by a source of unit strength located at $z_1 = 1$ and a sink of unit strength located at $z_2 = i$. Show that $F(z) = \log[(z^2 - 1)/(z^2 + 1)]$ is the complex potential for the flow shown in Figure 10.99.

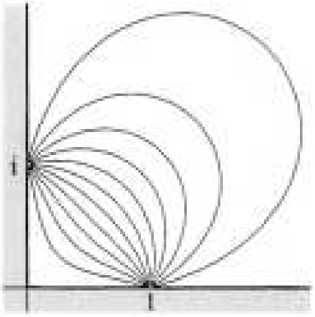

FIGURE 10.99 Accompanies Exercise 1.

2. Let the coordinate axes be walls of a containing vessel for a fluid flow in the first quadrant that is produced by two sources of equal strength located at the points $z_1 = 1$ and $z_2 = i$. Find the complex potential $F(z)$ for the flow in Figure 10.100.
3. Let the lines $x = 0$ and $x = \pi/2$ form the walls of a containing vessel for a fluid flow in the infinite strip $0 < x < \pi/2$ that is produced by a single source located at the point $z_0 = 0$. Find the complex potential for the flow in Figure 10.101.

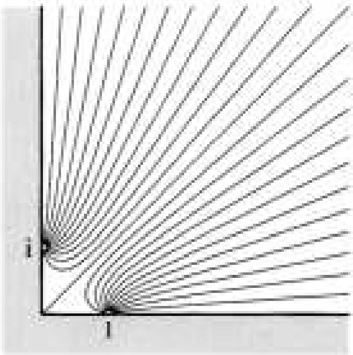

FIGURE 10.100 Accompanies Exercise 2.

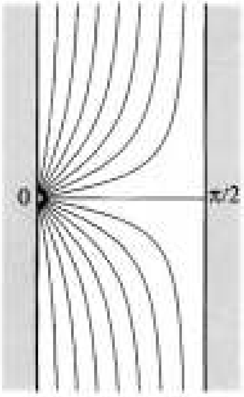

FIGURE 10.101 Accompanies Exercise 3.

4. Let the rays $x = 0$, $y > 0$ and $x = \pi$, $y > 0$ and the segment $y = 0$, $0 < x < \pi$ form the walls of a containing vessel for a fluid flow in the semi-infinite strip $0 < x < \pi$, $y > 0$ that is produced by two sources of equal strength located at the points $z_1 = 0$ and $z_2 = \pi$. Find the complex potential for the flow shown in Figure 10.102. *Hint*: Use the fact that $\sin(\pi/2 + z) = \sin(\pi/2 - z)$.

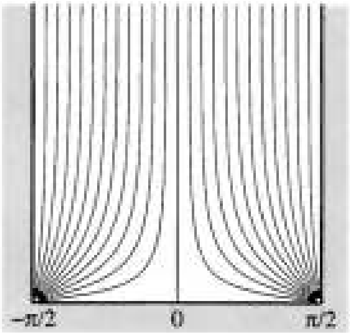

FIGURE 10.102 Accompanies Exercise 4.

· 5. Let the y axis be considered a wall of a containing vessel for a fluid flow in the right half plane $\mathrm{Re}(z) > 0$ that is produced by a single source located at the point $z_0 = 1$. Find the complex potential for the flow shown in Figure 10.103.

6. The complex potential $F(z) = 1/z$ determines an electrostatic field that is referred to as a dipole.

(a) Show that

$$F(z) = \lim_{a \to 0} \frac{\log(z) - \log(z - a)}{a}$$

and conclude that a dipole is the limiting case of a source and sink.

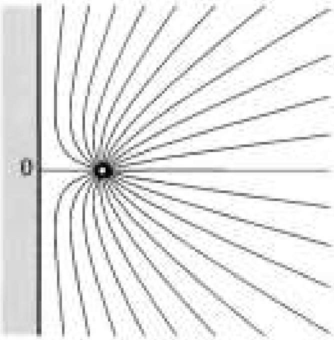

FIGURE 10.103 Accompanies Exercise 5.

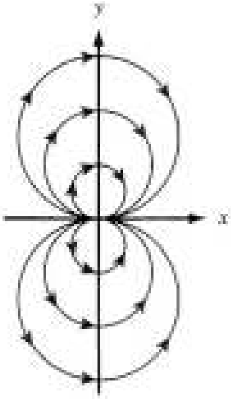

FIGURE 10.104 Accompanies Exercise 6.

(b) Show that the lines of flux of a dipole are circles that pass through the origin as shown in Figure 10.104.

7. Use a Schwarz-Christoffel transformation to find a conformal mapping $w = S(z)$ that will map the flow in the upper half plane onto the flow from a channel into a quadrant as indicated in Figure 10.105.

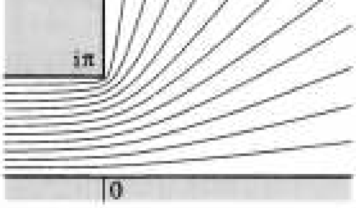

FIGURE 10.105 Accompanies Exercise 7.

8. Use a Schwarz-Christoffel transformation to find a conformal mapping $w = S(z)$ that will map the flow in the upper half plane onto the flow from a channel into a sector as indicated in Figure 10.106.

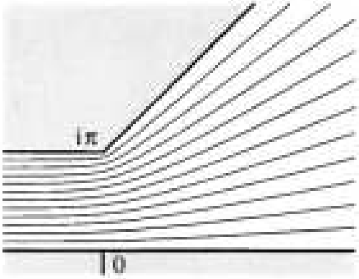

FIGURE 10.106 Accompanies Exercise 8.

9. Use a Schwarz-Christoffel transformation to find a conformal mapping $w = S(z)$ that will map the flow in the upper half plane onto the flow in a right-angled channel indicated in Figure 10.107.

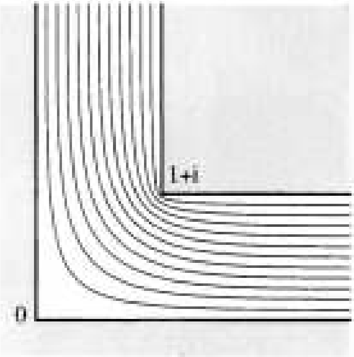

FIGURE 10.107 Accompanies Exercise 9.

10. Use a Schwarz-Christoffel transformation to find a conformal mapping $w = S(z)$ that will map the flow in the upper half plane onto the flow from a channel back into a quadrant as indicated in Figure 10.108, where $w_0 = 2\sqrt{2} - 2\ln(\sqrt{2} - 1) + i\pi$.

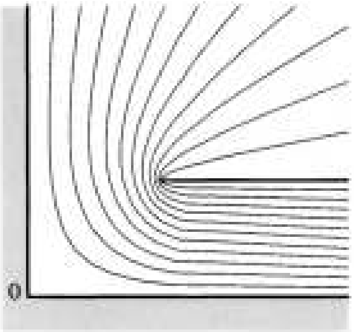

FIGURE 10.108 Accompanies Exercise 10.

11. **(a)** Show that the complex potential $F(z) = w$ given implicitly by $z = w + e^{w}$ determines the ideal fluid flow through an open channel bounded by the rays

$$y = \pi, -\infty < x < -1 \quad \text{and} \quad y = -\pi, -\infty < x < -1$$

into the plane.

(b) Show that the streamline $\psi(x, y) = c$ of the flow is given by the parametric equations

$$x = t + e^{t}\cos c, \quad y = c + e^{t}\sin c \quad \text{for } -\infty < t < \infty$$

as shown in Figure 10.109.

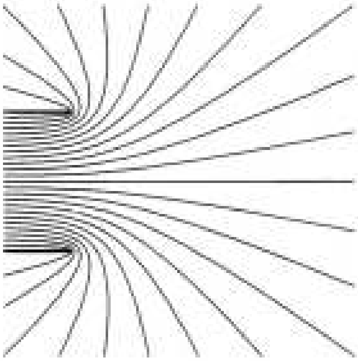

FIGURE 10.109 Accompanies Exercise 11.

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11. $U(t)$, given in Figure 11.11.

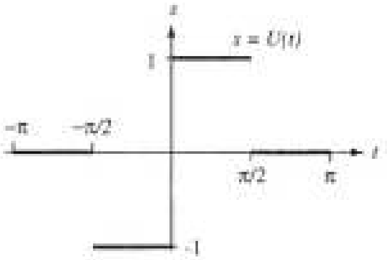

FIGURE 11.11 The graph of $U(t)$ for Exercise 11.

11.2 The Dirichlet Problem for the Unit Disk

The Dirichlet problem for the unit disk D: $|z| < 1$ is to find a real-valued function $u(x, y)$ that is harmonic in the unit disk D and that takes on the boundary values

(1) $u(\cos\theta, \sin\theta) = U(\theta)$ for $-\pi < \theta \le \pi$,

at points $z = (\cos\theta, \sin\theta)$ on the unit circle, as shown in Figure 11.12.

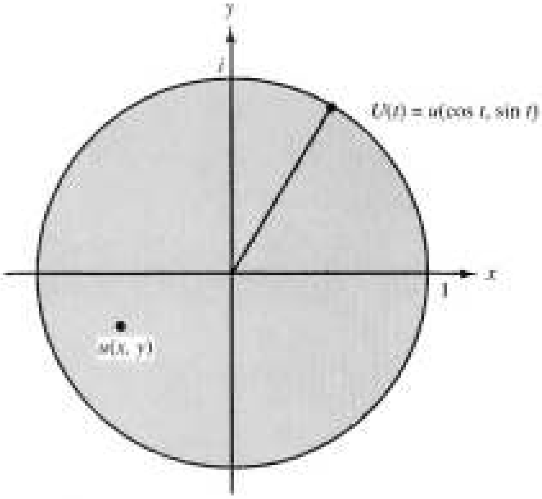

FIGURE 11.12 The Dirichlet problem for the unit disk $|z| < 1$.

Theorem 11.7 *If $U(t)$ has period 2π, and has the Fourier series representation*

(2) $$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt),$$

then the solution u to the Dirichlet problem in D is

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where $\{a_n\}$ and $\{b_n\}$ are the Fourier series coefficients for $U(t)$. This establishes the representation for $u(r\cos\theta, r\sin\theta)$ stated in Theorem 11.7.

EXAMPLE 11.3 Find the function $u(x, y)$ that is harmonic in the unit disk $|z| < 1$ and takes on the boundary values

$$(12)\quad u(\cos\theta, \sin\theta) = U(\theta) = \frac{\theta}{2} \quad \text{for } -\pi < \theta < \pi.$$

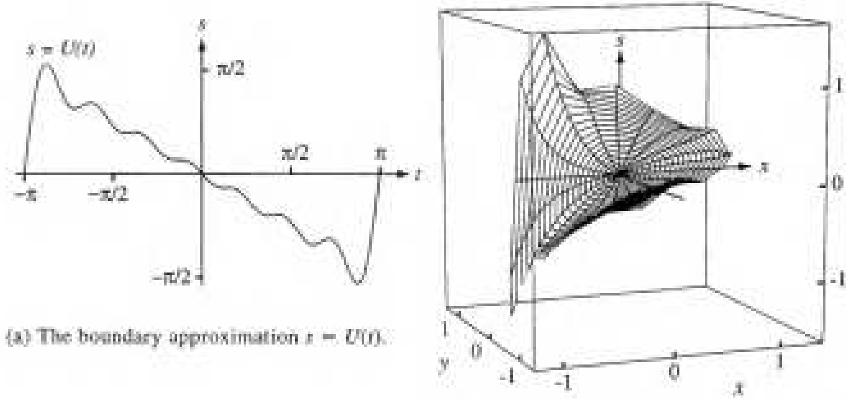
Solution Using Example 11.1, we write

$$(13)\quad U(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Using formula (3) for the solution of the Dirichlet problem, we obtain

$$(14)\quad u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin n\theta.$$

We remark that the series representation (14) for u takes on the prescribed boundary values (12) at points where U is continuous. The boundary function U is discontinuous at $z = -1$, which corresponds to $\theta = \pm\pi$; and U was not prescribed at these points. Graphs of the approximations $U_7(t)$ and $u_7(x, y) = u_7(r\cos\theta, r\sin\theta)$ which involve the first seven terms in equations (13) and (14), respectively, are shown in Figure 11.13.



(a) The boundary approximation $s = U(t)$.

(b) The graph of $s = u_7(r\cos\theta, r\sin\theta)$.

FIGURE 11.13 Functions $U_7(t)$ and $u_7(r\cos\theta, r\sin\theta)$ for Example 11.3.

EXERCISES FOR SECTION 11.2

For problems 1–6, find the solution to the given Dirichlet problem in the unit disk D by using the Fourier series representations for the boundary functions that were derived in the examples and exercises of Section 11.1.

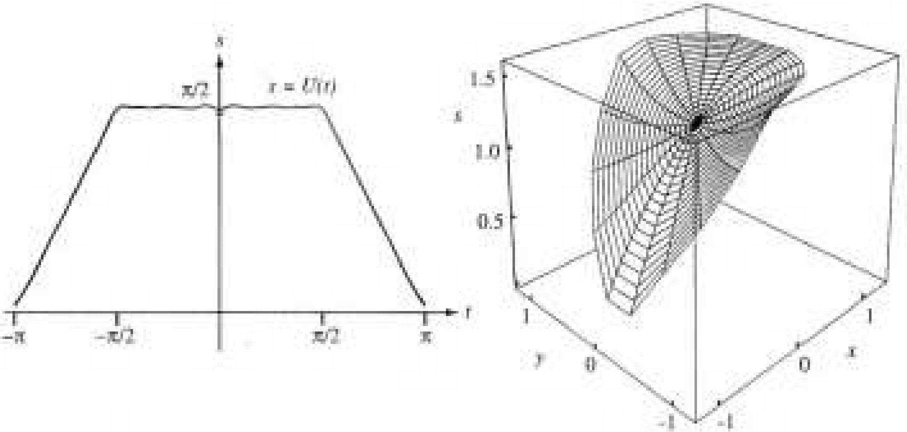


FIGURE 11.15 Approximations for $U_5(\theta)$ and $u_5(r\cos\theta, r\sin\theta)$ in Exercise 5.

6. $U(\theta) = \begin{cases} 1 & \text{for } \dfrac{\pi}{2} < \theta < \pi, \\ 0 & \text{for } \dfrac{-\pi}{2} < \theta < \dfrac{\pi}{2}, \\ -1 & \text{for } -\pi < \theta < \dfrac{-\pi}{2}. \end{cases}$ See Figure 11.16.

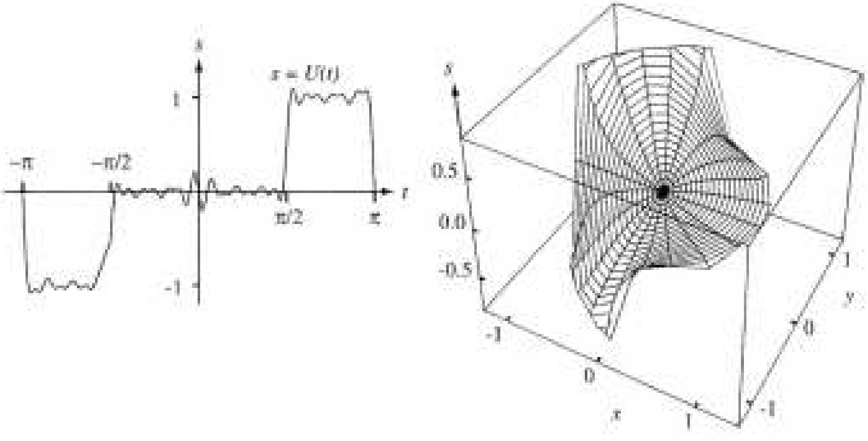


FIGURE 11.16 Approximations for $U_7(\theta)$ and $u_7(r\cos\theta, r\sin\theta)$ in Exercise 6.

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The last expression can be written as

$$(7)\quad F(s)G(s) = \int_0^\infty \left[\int_0^t f(t-\tau)g(\tau)\,d\tau \right] e^{-st}\,dt$$
$$= \mathcal{L}^{-1}\left(\int_0^t f(t-\tau)g(\tau)\,d\tau \right),$$

which establishes the proof of equation (2). Since we can interchange the role of the functions $f(t)$ and $g(t)$, equation (1) follows immediately.

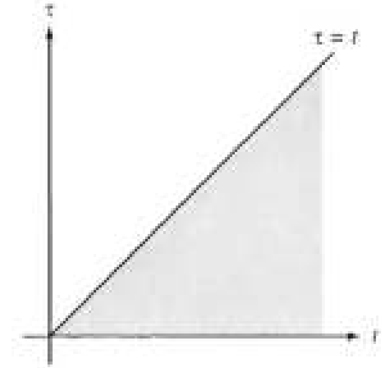

FIGURE 11.28 The region of integration in the convolution theorem.

TABLE 11.4 Properties of Convolution

Commutative	$f * g = g * f$
Distributive	$f * (g + h) = f * g + f * h$
Associative	$(f * g) * h = f * (g * h)$
Zero	$f * 0 = 0$

EXAMPLE 11.29 Show that $\mathcal{L}^{-1}\left(\dfrac{2s}{(s^2+1)^2}\right) = t \sin t$.

Solution Let $F(s) = \dfrac{1}{s^2+1}$, $G(s) = \dfrac{2s}{s^2+1}$, $f(t) = \sin t$, $g(t) = 2\cos t$, respectively. Applying the convolution theorem we get

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\frac{2s}{s^2+1}\right) = \mathcal{L}^{-1}(F(s)G(s)) = \int_0^t 2\sin(t-\tau)\cos\tau\,d\tau$$
$$= \int_0^t [2\sin t\cos^2\tau - 2\cos t\sin\tau\cos\tau]\,d\tau$$
$$= \sin t\,(\tau + \sin\tau\cos\tau) - \cos t\sin^2\tau \Big|_{\tau=0}^{\tau=t}$$
$$= t\sin t + \sin^2 t\cos t - \cos t\sin^2 t = t\sin t.$$

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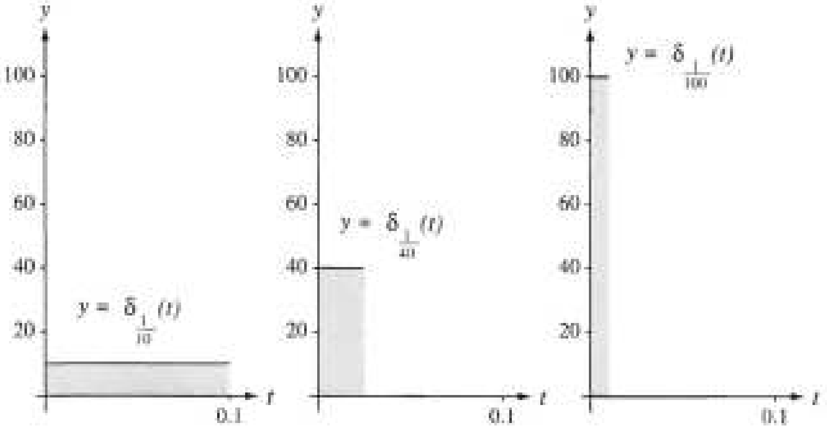

FIGURE 11.29 Graphs of $y = \delta_a(t)$ for $a = 10$, 40, and 100.

EXAMPLE 11.31 Show that $\mathcal{L}(\delta(t)) = 1$.

Solution By definition, the Laplace transform of $\delta_a(t)$ is

$$(10)\quad \mathcal{L}(\delta_a(t)) = \int_0^\infty \delta_a(t)e^{-st}\,dt = \int_0^a \frac{1}{a}e^{-st}\,dt = \frac{1-e^{-sa}}{sa}.$$

Letting $a \to 0$ in equation (10), and using L'Hôpital's rule, we obtain

$$(11)\quad \mathcal{L}(\delta(t)) = \lim_{a\to 0}\mathcal{L}(\delta_a(t)) = \lim_{a\to 0}\frac{1-e^{-sa}}{sa} = \lim_{a\to 0}\frac{se^{-sa}}{s} = 1.$$

We now turn our attention to the unit impulse function. First, consider the function $f_a(t)$ obtained by integrating $\delta_a(t)$:

$$(12)\quad f_a(t) = \int_0^t \delta_a(\tau)\,d\tau = \begin{cases} 0 & \text{for } t < 0, \\ \dfrac{t}{a} & \text{for } 0 \le t \le a, \\ 1 & \text{for } a < t. \end{cases}$$

Then it is easy to see that $U_0(t) = \lim_{a\to 0} f_a(t)$ (see Figure 11.30).

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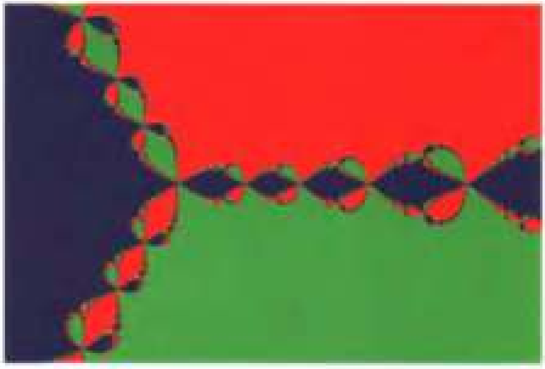
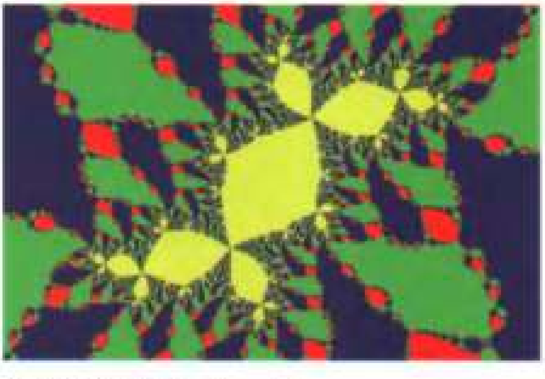
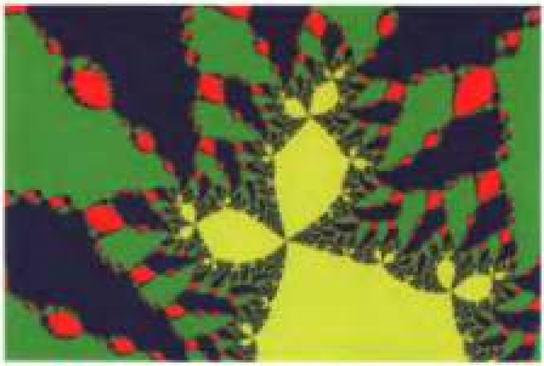


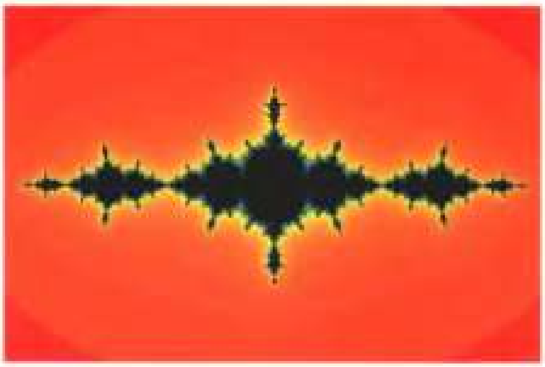
COLOR PLATE 1 Newton's method applied to $f(z) = z^2 + 1$.



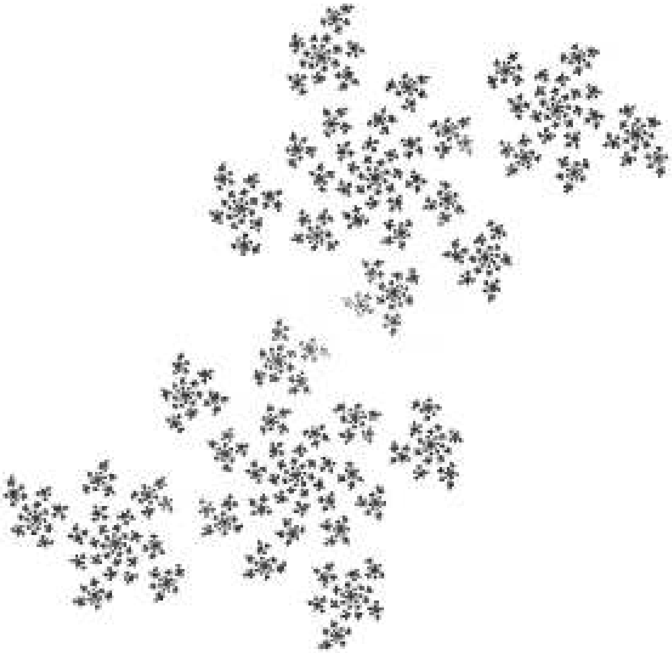
COLOR PLATE 2 The rabbit.



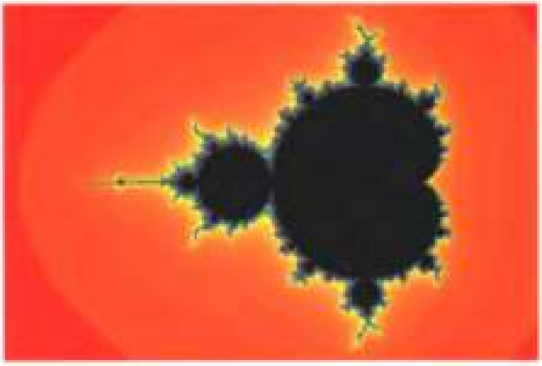
COLOR PLATE 3 A zoom of the rabbit.



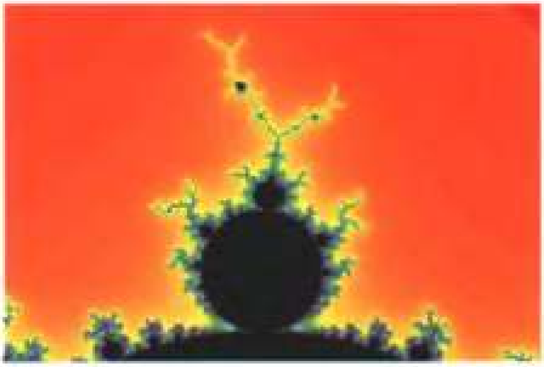
COLOR PLATE 4 The Julia set for $f(z) = z^2 - 1.25$.



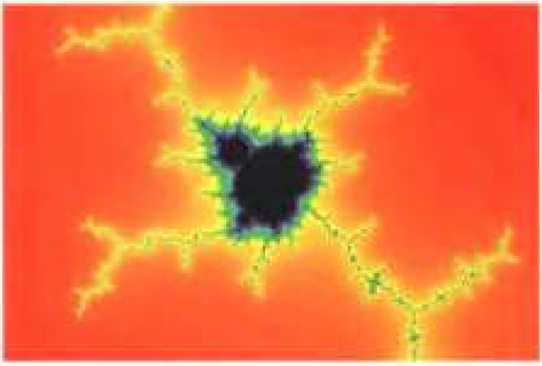
COLOR PLATE 5 A disconnected Julia set.



COLOR PLATE 6 The Mandelbrot set (M).



COLOR PLATE 7 A zoom on the upper portion of M.



COLOR PLATE 8 A zoom on the upper portion of color plate 7.

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